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Random Event Structures*

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Abstract In a line of recent development, probabilistic constructions of universal, homogeneous objects have been provided in various categories of ordered structures, such as causal sets[12], bifinite domains[13], and countable partial orders[10]. These constructions have been shown to produce objects with the desired properties with probability 1 in an appropriately defined measure space. A common strategy for these constructions is successive point-wise extension of an existing finite structure, with decisions on the relationships between the newly added point and the existing structure made according to well-specified probabilistic choices. This strategy is a departure from (and understandably so due to the increased complexity) the original one for random graphs[16] where a universal homogeneous countable graph is constructed with probability 1 in a single step (i.e., a single round of countably many probabilistic choices made independently). It would be interesting to see which of the categories studied more recently may admit such "one-step" constructions. The main focus of this paper is a new strategy, consisting of a single round of countably many probabilistic choices made independently, for the construction of a universal, homogeneous prime event structure. The intuition that the one-round construction is desirable has a similar flavor to a more general setting in e.g. Calculus/Real Analysis. When taking limits, iterative step-by-step processes are usually given, but a set of machineries was invented to determine the limit, i.e., achieving a "one-round" direct and explicit description of the limit.

Key words: event structure; universal domain; randomness; probability and order

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1 Introduction

The study of random structures began with Erdős and Rényi[16]'s remarkable probabilistic construction of countable graphs which, with probability 1, produces a universal homogeneous graph (called the random graphs). This and similar results by Erdős and Rényi[14,15] not only spurred further research on probabilistic laws in finite model theory, but also provided much foundational support for a flurry of recent high-profile activities in social networking research[1,4,7,22,28].

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An interesting property of a universal homogeneous object is that not only all finite structures in the background category can be embedded into it, but also any such two embeddings can be transformed into each other by an automorphism of the universal object (and hence it is called homogeneous). The investigation of universal structures can be traced back to Cantor[6] who showed that the chain $(\mathbb{Q}, \leqslant)$ is both universal and homogeneous in the class of countable linear orders. Later, Fraïssé[18] showed that the class of all countable undirected graphs contains a universal homogeneous object.

Within semantic frameworks for programming languages, the existence and construction of universal homogeneous objects have been investigated by a number of authors since the inception of domain theory. For example, Scott[26] provided a universal domain for the class of all ω-algebraic lattices and showed that in this domain calculations can be handled by a calculus of retracts; Plotkin[24] and Scott[27] studied universal domains for the classes of all coherent, respectively bounded-complete, ω-algebraic domains; and Gunter and Jung[20] and Droste and Göbel[9] described a systematic way of constructing universal and homogeneous domains. In particular, the work of Droste and Göbel[9] provides an explicit and deterministic construction of universal, homogeneous prime event structures which sets the background for the work presented in this paper. These structures have been shown to be useful in the study of currency and causality[29−33] (note that the notion of "probabilistic event structures" studied in Ref.[9] is different from ours).

In a sense, the investigation of probabilistic constructions of universal, homogeneous objects in various categories of domains is a natural step getting "back to the roots" planted in the work of Erdős and Rényi[16]. Of course, these richer structures present non-trivial challenges that need to be overcome in order to tame the increased complexity. This was indeed possible so far, as can be seen from the recent work of Droste and Kuske on causal sets[12], bifinite and Scott domains[13], and countable partial orders[10]. These constructions have been shown to produce objects with the desired properties with probability 1 in an appropriately defined probability space. A common strategy for these constructions is successive point-wise extension of an existing finite structure, with decisions on the relationships between the newly added point and the existing structure made according to well-specified probabilistic choices. This strategy is a departure from (and understandably so due to the increased complexity) the original one for random graphs[16] where the universal homogeneous countable graph is constructed with probability 1 in a single step (i.e., a single round of countably many probabilistic choices made independently). It would be interesting to see which of the categories studied more recently may admit such "one-step" constructions.

This paper introduces a new construction of universal, homogeneous prime event structures using a *single round of countably many probabilistic choices made independently*. In doing so, we hope that such a construction not only simplifies the process and the associated proofs, but also may reveal additional insights on the properties about universal homogeneous random prime event structures. For example, in Ref.[12] a probabilistic construction of universal homogeneous causal sets was given. Causal sets are partially ordered sets which have been proposed as a basic model for discrete space-time in quantum gravity. Since causal sets are a special class of prime event structures, the present result may shed new light into the "chaotic" and yet highly symmetric nature of these structures (the idea of emerging "Spontaneous Order"[28] is intriguing). We also discuss a similar construction in the closely related category of dI-domains.

2 Event Structures

Event structures[31] are mathematical structures for the study of concurrent computation through configurations enabled by the causality of events. An event structure is a description of a set of atomic actions in terms of a consistency and an enabling relation. The consistency relation indicates whether some events can occur together or not, and the enabling relation specifies the condition when a particular event may occur with regards to the occurrence of other events. We focus on prime event structures[31,32,33] in this paper, noting that the category of prime event structures **PES** is equivalent to the category of stable event structures, which is in turn equivalent to the category of dI-domains[33], using notations related to substructures and rigid embeddings[2]. In prime event structures, causality is modeled by an enabling relation in the form of a partial order. Events in the past of (i.e. smaller than) an event e are considered as causally necessary for e.

Definition 2.1 *A prime event structure is a triple*

$$\mathsf{E} = (E,\ Con, \leqslant)$$

where

- *E is a countable set of events,*
- *Con is a non-empty subset of* $\mathrm{Fin}(E)$ *(the collection of all finite subsets of E) called the consistency predicate which is required to be closed under $\subseteq$:*

$$X \subseteq Y \ \&\ Y \in Con \Rightarrow X \in Con,$$

- *$\leqslant$ is a finitary partial order on E, in that the down closure $\downarrow e := \{x \in E \mid x \leqslant e\}$ for any $e \in E$ is finite (this is also called the past-finite property).*

A configuration for such a structure is a consistent down-closed set.

It is reasonable to require that the consistency predicate Con interacts with the causality relation $\leqslant$ in such a way that if $X \in Con$, then $\downarrow X \in Con$, where $\downarrow X := \bigcup_{e\in X} \downarrow e$. However, this requirement does not affect the set of configurations determined by a prime event structure, and we do not make this requirement here.

2.1 Substructures

Definition 2.2 *For prime event structures $\mathsf{E}_1 = (E_1, Con_1, \leqslant_1)$ and $\mathsf{E}_2 = (E_2, Con_2, \leqslant_2)$, we say that E_1 is a substructure of E_2 and write $\mathsf{E}_1 \trianglelefteq \mathsf{E}_2$ if*

- *$\downarrow E_1 \subseteq E_2$, i.e., E_1 is down-closed in E_2,*
- *$Con_1 = \mathrm{Fin}(E_1) \cap Con_2$,*

Definition 2.3 *For prime event structures* $\mathsf{E}_1 = (E_1, Con_1, \leqslant_1)$ *and* $\mathsf{E}_2 = (E_2, Con_2, \leqslant_2)$*, we say that* E_2 *is an increment of* E_1 *and write* $\mathsf{E}_1 \prec \mathsf{E}_2$ *if* $\mathsf{E}_1 \trianglelefteq \mathsf{E}_2$ *and* $E_2 = E_1 \cup \{e\}$ *for some* $e \notin E_1$.

Note that if $\mathsf{E}_1 \prec \mathsf{E}_2$, then $X \in Con_2$ only if either $X \subseteq E_1$ and $X \in Con_1$, or else $X \setminus \{e\} \in Con_1$.

2.2 Universality and Homogeneity

Let us recall that a domain U of a class C of domains is called universal, if each other domain of C can be embedded (via an embedding-projection pair) into U, and U is homogeneous, if each isomorphism between two finite subdomains of U extends to an automorphism of U; intuitively, homogeneity means that U has the highest possible degree of structural symmetry.

To make the paper self-contained, we recall briefly a result of Droste and Göbel[9] concerning the existence of a universal, homogeneous object in an algebroidal category. Let $\mathbf{G}$ be a category in which all the morphisms are monic, and $\mathbf{G}^*$ a full subcategory of $\mathbf{G}$. Individually, an object U of $\mathbf{G}$ is called

- $\mathbf{G}^*$*-universal* if for any object A in $\mathbf{G}^*$, there is a morphism $f : A \to U$;
- $\mathbf{G}^*$*-homogeneous* if for any A in $\mathbf{G}^*$ and any pair $f, g : A \to U$, there is an isomorphism $h : U \to U$ such that $f = h \circ g$;

Intuitively, $\mathbf{G}^*$-homogeneity means that each isomorphism between two $\mathbf{G}^*$-substructures of U extends to an automorphism of U.

Collectively, the category $\mathbf{G}^*$ is said to have the *amalgamation property* if for any $f_1 : A \to B_1$, $f_2 : A \to B_2$ in $\mathbf{G}^*$, there exist $g_1 : B_1 \to B$, $g_2 : B_2 \to B$ in $\mathbf{G}^*$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Definition 2.4 *Let* $\mathbf{G}$ *be a category in which all morphisms are monic. Then* $\mathbf{G}$ *is called* algebroidal, *if* $\mathbf{G}$ *has the following properties:*

(1) $\mathbf{G}$ *has a weakly initial object,*

(2) *Every object of* $\mathbf{G}$ *is a colimit of an* ω*-chain of finite objects,*

(3) *Every* ω*-sequence of finite objects has a colimit, and*

(4) *The number (up to isomorphism) of finite objects of* $\mathbf{G}$ *is countable and between any pair of finite objects there exist only countably many morphisms.*

Theorem 2.1 *(Droste and Göbel[9]) Let* $\mathbf{G}$ *be an algebroidal category with all morphisms monic. Let* $\mathbf{G}_f$ *be the full subcategory of finite objects of* $\mathbf{G}$*. Then there exists a* $\mathbf{G}$*-universal,* $\mathbf{G}_f$*-homogeneous object iff* $\mathbf{G}_f$ *has the amalgamation property. Moreover, in this case the* $\mathbf{G}$*-universal,* $\mathbf{G}_f$*-homogeneous object is unique up to isomorphism.*

Let **PES** be the category of prime event structures with morphisms $\varphi : \mathsf{E}_1 \to \mathsf{E}_2$ such that $\varphi : E_1 \to E_2$ is a monomorphism resulting in $(\varphi(E_1), \varphi(Con), \varphi(\leqslant)) \trianglelefteq \mathsf{E}_2$, i.e., φ is injective and preserves and reflects both the order relations and the consistency predicates, and $\varphi(E_1)$ is closed downwards in E_2.

Then **PES** is equivalent to the category of dI-domains with rigid embeddings[9,33].

Proposition 2.1 **PES** *contains a universal homogeneous object.*

Proof: Note that **PES** is an algebroidal category with the finite objects being the (set-theoretically) finite prime event structures. Hence it suffices to check that the amalgamation property holds for finite prime event structures. For this one can simply take the set-theoretic unions of the two event sets, the two orders and the two consistency predicates from the components of the given prime event structures. Then Theorem 2.1 implies the result.

With a similar proof as Theorem 2.1 (a) in Ref.[12], we have the following characterization of universal homogeneous objects in **PES**.

Proposition 2.2 *An object* $\mathsf{U} = (U, Con, \leqslant)$ *in* **PES** *is universal and homogeneous iff it realizes all increments of finite substructures of* U*, i.e. for any finite* $\mathsf{E}_1 = (E_1, Con_1, \leqslant_1)$ *and any increment* $\mathsf{E}_2 = (E_1 \cup \{y\}, Con', \leqslant')$*, if* $\mathsf{E}_1 \trianglelefteq \mathsf{U}$*, then there exists* $z \in U$ *such that* $(\mathrm{id}_{E_1} \cup \{(y, z)\}) : \mathsf{E}_2 \to \mathsf{U}$ *is a morphism in* **PES**.

3 Random Event Structures

In this section, we present our probabilistic construction of countable prime event structures. It is not an essential restriction to assume that the prime event structures considered are given on the set of natural numbers, i.e. of the form $(\mathbb{N}, Con, \preceq)$. Thus we have to determine the order relation $\preceq$ and the consistency predicate Con. For this, we assume that we are given two families of probabilities $(p_i)_{i\in\mathbb{N}}$ and $\{q_X \mid X$ a finite non-empty subset of $\mathbb{N}\}$ such that $p_i, q_X \in (0,1)$ for all i and X. We use the p_i to determine the order relations $i \prec j$ and the probabilities q_X to determine whether X is *"preconsistent"*. Then we will call a set X *consistent*, if all of its non-empty subsets are preconsistent.

Our construction will work for each sequence of numbers $(p_i)_{i\in\mathbb{N}}$ and each family q_X (X a finite non-empty subset of $\mathbb{N}$).

Order: For $i < j$ in $\mathbb{N}$, put $i \prec j$ with probability p_i and then take the reflexive, transitive closure which is again denoted $\preceq$.

Consistency: We put $\emptyset \in Con'$. Now let F be a finite non-empty subset of $\mathbb{N}$. Put $F \in Con'$ with probability $q_{F\setminus(\max F)}$, where max$F$ is the maximal element of F (in the natural order $\leqslant$ on $\mathbb{N}$). A member of Con' is called *preconsistent*. Let $F \in Con$ if $\forall F' \subseteq F$, $F' \in Con'$.

Then $(\mathbb{N}, Con, \preceq)$ is a prime event structure. $\preceq$ is past finite because $(\mathbb{N}, \leqslant)$ is.

We claim that with probability 1 this event structure is universal and homogeneous, with the requirement on the probabilities given explicitly shortly.

The following lemma from Analysis will be applied in the proof (twice).

Lemma 3.1 (Folklore) *Let* $(a_i)_{i\in\mathbb{N}}$ *be a sequence of numbers with* $a_i \in [0,1)$ *for each* $i \in \mathbb{N}$*. Then*

$$\prod_{i\in\mathbb{N}}(1 - a_i) > 0$$

if and only if $\sum_{i\in\mathbb{N}} a_i < \infty$.

Theorem 3.1 *If* $\sum_{j\geqslant 1} p_j < \infty$*, then with probability* 1*,* $(\mathbb{N}, Con, \preceq)$ *is a universal homogeneous prime event structure.*

Proof: By Prop.2.2, it suffices to show that with probability 1, $(\mathbb{N}, Con, \preceq)$ realizes all increments of finite sub-event structures. Note that there are, up to isomorphism, only countably many such increments $((A, Con_A, \leqslant_A), (B, Con_B, \leqslant_B))$ with $A \subseteq \mathbb{N}$. Since the intersection of countably many events of probability 1 again has probability 1, it suffices to consider an arbitrary fixed increment $(A, Con_A, \leqslant_A) \lhd (B, Con_B, \leqslant_B)$ with $A \cup \{y\} = B$ and $(A, Con_A, \leqslant_A) \lhd (\mathbb{N}, Con, \preceq)$. Then $A \subseteq A_m$ for some $m \in \mathbb{N}$, where $A_m := \{1, 2, \ldots, m\}$ is the mth initial segment of $\mathbb{N}$. We claim that then with probability 1, there exists $n \in \mathbb{N}$ with $n > m$ such that

$$f_n = (\mathrm{id}_A \cup \{(y, n)\}) : (B, Con_B, \leqslant_B) \to (\mathbb{N}, Con, \preceq)$$

is a **PES**-morphism. For this, we will show that the probability that for no $n > m$, f_n is a **PES**-morphism, equals 0.

To this aim, consider any integer $n \in \{m+1, m+2, \ldots\}$. We wish to compute a lower bound r_n for the probability that f_n is a **PES**-morphism. For this, f_n would have to preserve and reflect both the order relation and the consistency predicate, and $f_n(B)$ would have to be closed downwards in $\mathbb{N}$.

First let us consider the order relation. Since $(A, Con_A, \leqslant_A)$ is a substructure of $(\mathbb{N}, Con, \preceq)$, it suffices to check that for each $a \in A$, we have $a <_B y$ iff $a \prec n$. Since $\preceq$ is obtained as a transitive closure, the probability that we put $a \prec n$ in our construction is at least p_a for each $a \in A$ with $a <_B y$. If $a \not<_B y$, the event that $a \not\prec n$ breaks into two independent sub-events: for each $j \in A_n$, either $a \not\prec j$ or $j \not\prec n$. For this, we ensure that $j \not\prec n$ for each $j \in A_n \setminus A$ and for each $j \in A$ with $j \not<_B y$. The probability for this is at least $\prod_{j\in A_n}(1 - p_j)$.

We also have to ensure that $f(B)$ is closed downwards in $\mathbb{N}$. For this, we have to enforce that there is no $j \in A_n \setminus A$ with $j \prec n$. But this we already ensured in the previous paragraph.

Hence the probability that $a <_B y$ iff $a \prec n$ is at least

$$\prod_{a\in A} p_a \cdot \prod_{j\in A_n} (1 - p_j).$$

Second let us consider the consistency predicate. We get the equivalence $F \cup \{y\} \in Con_B \Leftrightarrow F \cup \{n\} \in Con$ for each $F \subseteq A$ certainly, if we assume that $F \cup \{y\} \in Con_B \Leftrightarrow F \cup \{n\} \in Con'$ for each $F \subseteq A$.

For such an F, the probability for this is at least $\min\{q_F, (1 - q_F)\}$. Hence the probability that f_n preserves and reflects consistency, i.e. that we make for all such F the right consistency decision, is at least

$$\prod_{F\subseteq A} \min\{q_F, (1 - q_F)\}.$$

This quantity depends only on A, and not on m or n, and (surprizingly) not even on $B = A \cup \{y\}$. The reason is that when we made the choice about preconsistency of $X \in Con'$, its probability $q_{X\setminus(\max X)}$ is independent of the maximal element of X.

Thus the probability that f_n is a **PES**-morphism is bounded from below by:

$$r_n := \left(\prod_{a\in A} p_a \cdot \prod_{j\in A_n} (1-p_j)\right) \cdot \left(\prod_{F\subseteq A} \min\{q_F, (1-q_F)\}\right).$$

Hence the probability that f_n is not the required morphism is at most $1-r_n$. Consequently, the probability that for *all* $n \in \{m+1,\dots\}$, the mapping f_n is not a **PES**-morphism is at most $\prod_n(1-r_n)$. Note that $\prod_n(1-r_n)=0$ iff $\sum_n r_n=\infty$ (by Lemma 3.1). Hence, if $\sum_{n\geqslant 1}\prod_{1\leqslant j\leqslant n}(1-p_j)=\infty$ then almost surely the increment is realized in $(\mathbb{N}, Con, \preceq)$. A sufficient condition for $\sum_{n\geqslant 1}\prod_{1\leqslant j\leqslant n}(1-p_j)=\infty$ is $\prod_{j\geqslant 1}(1-p_j)\neq 0$, which (again by Lemma 3.1) is equivalent to our assumption $\sum_{j\geqslant 1} p_j<\infty$.

Note that the conclusion of Theorem 3.1 also holds if we let $p_j := 1/j$ (for all $j\geqslant 2$), since then $\prod_{2\leqslant j\leqslant n}(1-p_j)=1/n$, so $\sum_n r_n=\infty$. If we let $p_j := p \in (0,1)$, a constant (for all $j\geqslant 1$), we have $\sum_n r_n<\infty$, and the analysis does not work any more.

Remark. Note that the assumption $\sum\limits_{j\geqslant 1} p_j < \infty$ in Theorem 3.1 is essential. In general, the outcome of the construction given at the beginning of this section depends very much on the given sequence $(p_i)_{i\in\mathbb{N}}$. As an illustration of an extreme case, we determine the probability that in the resulting event structure $(\mathbb{N}, Con, \preceq)$ the order relation is linear. Since we can only put $i \prec k$ iff $i<k$ in the natural order on $\mathbb{N}$, it follows that then the event structure order relation $\preceq$ coincides with the natural order $\leqslant$ on $\mathbb{N}$. Indeed, this happens if we put $i \prec i+1$ for each $i\in\mathbb{N}$. Since these decisions are independent of each other, the total probability for this equals $p=\prod_{i\in\mathbb{N}} p_i$. Hence, by Lemma 3.1, $(\mathbb{N}, \preceq)$ is a chain with positive probability iff $\sum_{i\in\mathbb{N}}(1-p_i)<\infty$. This is the case, for instance, if $p_1=1/2$ and $p_i=1-1/i^2$ for each $i\geqslant 2$.

4 A Metric Space of Prime Event Structures

In this section, we wish to endow the collection of all prime event structures over $\mathbb{N}$ with a metric, and we will show that in this metric space, the collection of all universal homogeneous prime event structures forms a "large" subset in a topological sense. We first recall some basic notions from topology, cf. e.g.[23]. Let (X,d) be a complete metric space. A subset S of X is *open* if for each $s\in S$ there is $\varepsilon>0$ such that the ε-ball $B_\varepsilon(s)=\{x\in X \mid d(s,x)<\varepsilon\}$ of s is contained in S. The set S is *dense* if its closure equals X, i.e., S meets every non-empty open set. A subset R is *residual* if R contains the intersection of countably many open dense sets (in equivalent terminology, R is the complement of a meagre set).

Proposition 4.1 (**Bairecategorytheorem**) *A residual set in a complete metric space is non-empty.*

Hence, if we can show that the collection of all universal homogeneous prime event strucures forms a residual subset of some complete metric space, it follows that there *exists* such a prime event structure. In fact, a residual set R is considered as "large" containing "almost all" of the space; for instance, R meets every open dense set, and the intersection of countably many residual sets is again residual. We refer the reader to Ref.[5] for applications of this in algebra.

We construct a metric space (X, d) as follows. Let

$$X = \{0,1\}^{\mathcal{P}_{\text{fin}}(\mathbb{N})} \times \{0,1\}^{\mathbb{N}^2} = \mathcal{P}(\mathcal{P}_{\text{fin}}(\mathbb{N})) \times \{R \mid R \subseteq \mathbb{N}^2\},$$

where $\mathcal{P}(X)$ stands for the powerset of X, and $\mathcal{P}_{\text{fin}}(X)$ stands for the set of finite subsets of X. Let $d((Con_1, \leqslant_1), (Con_2, \leqslant_2)) := 2^{-n}$ where (recall that $A_m = \{1, \ldots, m\}$)

$$n := \sup\{m \in \mathbb{N} \mid \ Con_1 \cap \mathcal{P}(A_m) = Con_2 \cap \mathcal{P}(A_m) \ \& \ \leqslant_1 \cap A_m^2 = \leqslant_2 \cap A_m^2\},$$

with $2^{-\infty} = 0$. The topology on X is determined by the basic open sets

$$\{(Con, R) \in X \mid Con \cap \mathcal{P}(A_m) = F \ \& \ R \cap A_n^2 = G\}$$

for finite sets F and G and $m, n \in \mathbb{N}$. Then it is well-known that (X, d) is a complete metric space, and the topology of X is compact.

We will call a prime event structure $(\mathbb{N}, Con, \preceq)$ *approximable*, if for each $n \geqslant 1$, $\mathsf{A}_n := (A_n, Con \cap \mathcal{P}(A_n), \preceq \cap {A_n}^2) \trianglelefteq (\mathbb{N}, Con, \preceq)$. Note that then we have a chain of increments $\mathsf{A}_1 \prec \mathsf{A}_2 \prec \mathsf{A}_3 \prec \ldots$. We note that approximability is not an essential restriction since for any countable prime event structure $\mathsf{E} = (E, Con, \preceq)$, we can enumerate $E = \{e_i \mid i \geqslant 1\}$ in such a way that $\mathsf{E}_n := (E_n, Con \cap \mathcal{P}(E_n), \preceq \cap {E_n}^2) \trianglelefteq (E, Con, \preceq)$, where $E_n := \{e_i \mid 1 \leqslant i \leqslant n\}$.

Now let $P_{\text{pes}} := \{(Con, \preceq) \mid (\mathbb{N}, Con, \preceq)$ is an approximable prime event structure$\}$.

Observe that if $(Con, \preceq) \in X \setminus P_{\text{pes}}$, then either Con is not closed under subsets, or $\preceq$ is not a partial order, or some A_n $(n \geqslant 1)$ is not down-closed in $(\mathbb{N}, \preceq)$. In each case, $(Con, \preceq)$ is contained in a basic open set disjoint to P_{pes}, showing that P_{pes} is a closed subset of (X, d) and hence a complete metric space.

Lemma 4.1 *Let* $\mathsf{A} = (A, Con_A, \leqslant_A)$ *and* $\mathsf{B} = (B, Con_B, \leqslant_B))$ *be an increment with* $A \subseteq \mathbb{N}$ *finite. Then*

$S_{\mathsf{A},\mathsf{B}} := \{(Con, \preceq) \in P_{\text{pes}} \mid$ if $\mathsf{A} \trianglelefteq (\mathbb{N}, Con, \preceq)$ then there exits **PES**-morphism $g : \mathsf{B} \to (\mathbb{N}, Con, \preceq)$ such that $g \restriction_A = \text{id}_A\}$

is an open and dense subset of P_{pes}.

Proof: First we show that $S_{\mathsf{A},\mathsf{B}}$ is open. It is easily seen that $S_{\mathsf{A},\mathsf{B}}$ is the union of the following open sets

- $\{(Con, \preceq) \in P_{\text{pes}} \mid \exists g : (B, Con_B, \leqslant_B) \to (A_n, Con \cap \mathcal{P}(A_n), \preceq \cap {A_n}^2)$ with $g \restriction_A = \text{id}_{\mathsf{A}}\}$, where g is a **PES**-morphism and $n \in \mathbb{N}$,

- $\{(Con, \preceq) \in P_{\text{pes}} \mid (A, Con, \leqslant_A) \ntrianglelefteq (A_m, Con \cap \mathcal{P}(A_m), \preceq \cap {A_m}^2)\}$, where $m \in \mathbb{N}$ is minimal with $A \subseteq A_m$.

Hence $S_{\mathsf{A},\mathsf{B}}$ is a union of open sets and therefore open.

Second, we show that $S_{\mathsf{A},\mathsf{B}}$ is dense, i.e., intersects any basic open set. For this purpose, let $(C, Con_C, \leqslant_C)$ be some finite prime event structure with $C \subseteq \mathbb{N}$. Let V be the basic open set determined by $(Con_C, \leqslant_C)$. We claim that $S_{\mathsf{A},\mathsf{B}} \cap V \neq \emptyset$. If there is no $(Con, \preceq)$ in P_{pes} such that $(A, Con_A, \leqslant_A), (C, Con_C, \leqslant_C) \trianglelefteq (\mathbb{N}, Con, \preceq)$, then $V \subseteq S_{A,B}$. Otherwise let $(Con, \preceq)$ be a member in P_{pes} such that $(A, Con_A, \leqslant_A)$, $(C, Con_C, \leqslant_C) \trianglelefteq (\mathbb{N}, Con, \preceq)$.

Since A, C are finite, there exists $n \in \mathbb{N}$ with $(A, Con_A, \leqslant_A), (C, Con_C, \leqslant_C) \trianglelefteq (A_n, Con_n, \leqslant_n)$, where $\leqslant_n = \preceq \cap {A_n}^2$ and Con_n is the restriction of Con to A_n. It suffices to show that $S_{\mathsf{A,B}}$ intersects the basic open set determined by $\leqslant_n$, i.e., that $S_{\mathsf{A,B}}$ contains some pair $(Con', \preceq')$ with $(A_n, Con_n, \leqslant_n) \trianglelefteq (\mathbb{N}, Con', \preceq')$. For this, let $B = A \cup \{y\}$. Then define $\leqslant_{n+1} \subseteq {A_{n+1}}^2$ and Con_{n+1} as follows:

- $\leqslant_{n+1} := \leqslant_n \cup \{(x, n+1)\} \mid x \in A, x <_B y\}$,
- $Con_{n+1} := Con_n \cup \{X \cup \{n+1\} \mid X \in Con_A, X \cup \{y\} \in Con_B\}$.

Then $(A_n, Con_n, \leqslant_n) \triangleleft (A_{n+1}, Con_{n+1}, \leqslant_{n+1})$. Furthermore, the mapping

$$g = \mathrm{id}_A \cup \{(y, n+1)\} : (B, Con_B, \leqslant_B) \to (A_{n+1}, Con_{n+1}, \leqslant_{n+1})$$

is a **PES**-morphism. Now extend $\leqslant_{n+1}$ trivially to a past-finite partial order, denoted $\preceq'$, on $\mathbb{N}$ and put $Con' := Con_{n+1}$, a consistency predicate on $\mathbb{N}$. Then g is a **PES**-morphism from $(B, Con_B, \leqslant_B)$ to $(\mathbb{N}, Con', \preceq')$, i.e., $(Con', \preceq') \in S_{A,B}$ and $(A, Con_A, \leqslant_A) \triangleleft (A_{n+1}, Con_{n+1}, \leqslant_{n+1}) \triangleleft (\mathbb{N}, Con', \preceq')$, as needed.

Theorem 4.1 *In the metric space* (P_{pes}, d)*, the subset*

$$S = \{(Con, \preceq) \in P_{\text{pes}} \mid (\mathbb{N}, Con, \preceq) \textit{ is universal and homogeneous}\}$$

is residual.

Proof: We claim that S contains the intersection of all sets $S_{A,B}$ where $((A, Con_A, \leqslant_A), (B, Con_B, \leqslant_B))$ is an increment in B. Indeed, if $(Con, \preceq)$ belongs to this intersection, then $(\mathbb{N}, Con, \preceq)$ realizes all increments of finite sub-event structures and hence is universal and homogeneous by Prop.2.2. Clearly, there are just countably many such sets $S_{\mathsf{A,B}}$, proving that S is residual.

5 Causal Sets

Causal sets[3] have been proposed as a basic model for discrete spacetime in quantum gravity theory. They are defined as partially ordered sets with the past-finite property (cf. Def. 2.1). Hence they are prime event structures where all events are consistent. The substructure relationship (a.k.a. stem) employed for causal sets is precisely the same as the one used here for prime event structures. If we restrict all of the previous considerations to prime event structures with a trivial consistency predicate, two corollaries are immediate consequences:

Corollary 5.1 *If* $\sum_{j \geqslant 1} p_j < \infty$*, then with probability* 1*,* $(\mathbb{N}, \preceq)$ *is a universal homogeneous causal set.*

This corollary says that universal homogeneous causal sets[12] can be constructed by a single round of countably many probabilistic choices made independently, with probability 1.

Corollary 5.2 *In the metric space of past-finite approximable partial orders over* $\mathbb{N}$*, the universal homogeneous ones form a subset which is residual.*

6 dI-domains

Finally, we wish to apply our main results to dI-domains, using a combination of ideas as given in section 3 and in Ref.[13].

Recall that a *Scott-domain* is a bounded-complete $\omega-$algebraic cpo. We let $K(D)$ denote the set of compact elements of D. A Scott-domain $(D, \sqsubseteq)$ is called a *dI-domain*, if for each $x \in K(D)$, the set $\{d \in D \mid d \sqsubseteq x\}$ is a finite distributive lattice. An element $p \in D$ is called a *complete prime*, if whenever $p \sqsubseteq \bigsqcup S$ for some subset $S \subseteq D$, then $p \sqsubseteq s$ for some $s \in S$. Let $Pr(D)$ denote the set of all complete primes of D.

Now let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be two dI-domains and $f : D \to D', g : D' \to D$ two continuous functions. Then (f, g) is called a *stable embedding-projection pair* from $(D, \sqsubseteq)$ to $(D', \sqsubseteq')$, if $g \circ f = id_D, f \circ g \leqslant id'_D$ (in the pointwise ordering of functions), and whenever $x \in D, y \in D'$ and $x \leqslant g(y)$, then $g \circ f(x) = x$. In the literature, then f is also often called a *rigid embedding*.

Now if E is a prime event structure, its domain $(D(\mathsf{E}), \subseteq)$ of configurations, ordered by inclusion, forms a dI-domain. Conversely, given a dI-domain $(D, \sqsubseteq)$, let Con comprise all finite subsets of $Pr(D)$ which are bounded above in $(D, \sqsubseteq)$. Then $(Pr(D), Con, \sqsubseteq)$ is a prime event structure whose domain of configurations is isomorphic to $(D, \sqsubseteq)$. As is well-known, this correspondence (together with associating to a stable embedding-projection pair (f, g) the restriction $f \restriction_{Pr(D)}$ as event structure morphism) induces a category equivalence between the category of all dI-domains and the category **PES** of all prime event structures. Therefore, when considering dI-domains $(D, \sqsubseteq)$, we may assume them to be given and uniquely determined (up to some natural completion process) by the structure of their associated prime event structure $(Pr(D), Con, \sqsubseteq)$. We refer the reader to Winskel[31,32] and Zhang[33] for further background on this.

It has been shown that the category of dI-domains contains a universal homogeneous object (Droste and Göbel[9]). Now we obtain a new construction of this domain:

Corollary 6.1 *Let* $(p_i)_{i \in \mathbb{N}} \in (0, 1)$ *such that* $\sum_{i \geqslant 1} p_i < \infty$, *and let* $(q_X \mid X \subseteq \mathbb{N}$ and X is finite and non-empty$)$ *be a family with* $q_X \in (0, 1)$ *for all* $X \subseteq \mathbb{N}$. *Construct the event structure* $\mathsf{U} = (\mathbb{N}, Con, \preceq)$ *randomly as in section 3. Then with probability 1,* $(D(\mathsf{U}), \subseteq)$ *is the universal homogeneous dI-domain.*

Proof: Immediate by Theorem 3.1 and the category equivalence described above.

We also obtain a corresponding topological result as follows. Let us call a dI-domain $(D, \sqsubseteq)$ *approximable*, if $Pr(D) = \mathbb{N}$ and for each $n \in \mathbb{N}, A_n$ is down-closed in $\mathbb{N}$ (with respective to the order given on D). Clearly, each infinite dI-domain $(D, \sqsubseteq)$ is isomorphic to an approximable one. If $(D, \sqsubseteq)$ is approximable, then $(Pr(D), Con, \sqsubseteq)$ is an approximable event structure. Hence by section 4, we can endow the collection dI_{approx} of all approximable dI-domains with a metric making it isomorphic to the metric space (P_{pes}, d). As an immediate consequence of Theorem 4.1 and the category equivalence described above we obtain:

Corollary 6.2 *In the complete metric space* (dI_{approx}, d), *the collection of all* *universal and homogeneous dI-domains is residual.*

7 Conclusion

Prime event structures have been shown to be useful models for concurrency and for sequential computation. Together with stable functions, they form a cartesian closed categoryF[31,32], which exemplifies their structural richness. We have shown in this paper that there exists a countable prime event structure U which is universal, i.e. it contains any other countable prime event structure as a substructure (up to isomorphism). Moreover, U can be built to be homogeneous, i.e. (in a precise mathematical sense) highly symmetrical. With these two properties, U is unique up to isomorphism.

Our construction provides a range of parameters each of which induces a probability space of prime event structures on $\mathbb{N}$. Our main result shows, that under fairly weak assumptions on these parameters, in each of these probability spaces almost any prime event structure is universal and homogeneous, i.e., the collection of all universal homogeneous prime event structures has measure 1. We have also shown that the collection of prime event structures on $\mathbb{N}$ can be endowed with (in a standard way) the structure of a complete metric space, and in this space the collection of all universal homogeneous prime event structures is topologically large: it is the complement of a meager set.

Similar constructions in the categories of causal sets and dI-domains are also presented.

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