Int J Software Informatics, Vol.1, No.1, December 2007, pp. 129-152 E-mail: ijsi@iscas.ac.cn
International Journal of Software and Informatics ISSN 1673-7288 http://www.ijsi.org

An Institutional View on Categorical Logic

Joseph Goguen[1], Till Mossakowski[2], Valeria de Paiva[3], Florian Rabe[4],
Lutz Schröder[2]

[1](University of California, San Diego, USA)
[2](DFKI-Lab Bremen and University of Bremen, Germany)
[3](Palo Alto Research Center, Palo Alto, California, USA)
[4](Jacobs University Bremen, Germany)

Abstract We introduce a generic notion of categorical propositional logic and provide a construction of a preorder-enriched institution out of such a logic, following the Curry-Howard-Tait paradigm. The logics are specified as theories of a meta-logic within the logical framework LF such that institution comorphisms are obtained from theory morphisms of the meta-logic. We prove several logic-independent results including soundness and completeness theorems and instantiate our framework with a number of examples: classical, intuitionistic, linear and modal propositional logic.

Key words: categorical logic; propositional logic; institutions; logic translations; Curry-Howard-Tait isomorphism

Goguen J, Mossakowski T, de Paiva V, Rabe F, Schöder L. An institutional view on categorical logic. *Int J Software Informatics*, **2007, 1(1): 129-152. http://www.ijsi.org/1673-7288/1/127.pdf**

We dedicate this work to the memory of our dear friend and colleague Joseph Goguen who passed away during its preparation.

1 Introduction

The well-known Curry-Howard isomorphism[25,18] establishes a correspondence between propositions and types, and proofs and terms, and proof and term reductions. A number of correspondences between various kinds of logical theories, λ-calculi and categories have been established, see Fig. 1.

Here, we present work aimed at casting the propositional part of these correspondences in a common framework based on the theory of *institutions*. The notion of institution arose within computer science as a response to the population explosion among logics in use[20,22]. Its key idea is to focus on abstractly axiomatizing the *satisfaction* relation between sentences and models. A wide range of logics, e.g., equa-

Corresponding author: Till Mossakowski, Till.Mossakowski@dfki.de
Manuscript received 1 Oct. 2007; revised 27 Nov. 2007; accepted 16 Dec. 2007; published online 29 Dec. 2007.

conjunctive logic	pairing	cartesian categories	[26]
positive logic	λ with paring	cartesian closed categories	[26]
intuitionistic propositional logic	λ with pairing and sums	bicartesian closed categories	[26]
classical propositional logic	λ with pairing, sums and $\neg\neg$-elimination	bicartesian closed categories with $\neg\neg$-elimination	[26]
modal logics, e.g., intuitionistic S4		monoidal comonad with strong monad	[6]
linear logic		$*$-autonomous categories	[39]
first-order logic	λ with pairing, sums and certain dependent types	hyperdoctrines	[37]
Martin-Löf type theory	λ with pairing, sums and dependent types	locally cartesian closed categories	[38]

Figure 1. Curry-Howard correspondences

tional[20], Horn[22], first-order[20], modal[8,40], higher-order[7], polymorphic[36], temporal[16], process[16], behavioral[5], and object-oriented[21] logics, have been formalized as institutions. A surprisingly large amount of meta-logical reasoning can be carried out in this abstract framework; e.g., institutions have been used to give general foundations for modularization of theories and programs[14,35], and substantial portions of classical model theory can be lifted to the level of institutions[12,9,10,42]. The objective is to be able to apply each meta-mathematical result to the widest possible range of abstract institutions; see e.g. [9, 10, 11, 12, 13].

In the sequel, we give a general notion of categorical propositional logic, and a construction of a proof-theoretic institution out of such a logic that formalizes the abstract Curry-Howard-Tait correspondence. Then we use the institutional meta-theory to establish logic-independent results including general soundness and completeness theorems, and we discuss how the Curry-Howard-Tait isomorphism can be cast as a morphism of institutions.

2 Institutions and Logics

We assume that the reader is familiar with basic notions from category theory (cf. e.g.[1,27]). We denote the class of objects of a category $\mathbb{C}$ by $|\mathbb{C}|$, the set of morphisms form A to B by $\mathbb{C}\ (A, B)$, and composition by o. Let $\mathbb{CAT}$ be the the category of all categories (which of course lives in a higher set-theoretic universe). Institutions are defined as follows.x

Definition 2.1. An *institution* $\mathcal{I} = (\mathbb{S}ign^{\mathcal{I}}, \mathsf{Sen}^{\mathcal{I}}, \mathsf{Mod}^{\mathcal{I}}, \models^{\mathcal{I}})$ consists of

• a category $\mathbb{S}ign^{\mathcal{I}}$ of *signatures*;

• a functor $\mathsf{Sen}^{\mathcal{I}} : \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}et$ giving, for each signature Σ, the set $\mathsf{Sen}^{\mathcal{I}}(\Sigma)$ of *sentences*, and for each signature morphism σ: $\Sigma \to \Sigma'$, the *sentence translation map* $\mathsf{Sen}^{\mathcal{I}}(\sigma) : \mathsf{Sen}^{\mathcal{I}}(\Sigma) \to \mathsf{Sen}^{\mathcal{I}}(\Sigma')$, with $\sigma(\varphi)$ abbreviating $\mathsf{Sen}^{\mathcal{I}}(\sigma)(\varphi)$;

• a functor $\mathsf{Mod}^{\mathcal{I}} :\ (\mathbb{S}ign^{\mathcal{I}})^{op} \to \mathbb{CAT}$ giving, for each signature Σ, the category $\mathsf{Mod}^{\mathcal{I}}\ (\Sigma)$ of *models*, and for each signature morphism $\sigma : \Sigma \to \Sigma'$, the *reduct functor* $\mathsf{Mod}^{\mathcal{I}}(\sigma) : \to \mathsf{Mod}^{\mathcal{I}}(\Sigma') \to \mathsf{Mod}^{\mathcal{I}}(\Sigma)$, with $M'|_{\sigma}$ abbreviating $\mathsf{Mod}^{\mathcal{I}}(\sigma)(M')$, the σ-*reduct* of M'; and

• a satisfaction relation $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathsf{Mod}^{\mathcal{I}}(\Sigma)| \times \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$,

such that for each $\sigma : \Sigma \to \Sigma'$ in $\mathbb{S}ign^{\mathcal{I}}$, the *satisfaction condition*

$$M' \models^{\mathcal{I}}_{\Sigma'} \sigma(\varphi) \Leftrightarrow M'|_{\sigma} \models^{\mathcal{I}}_{\Sigma} \varphi$$

holds for all $M' \in |\mathsf{Mod}^{\mathcal{I}}(\Sigma')|$ and all $\varphi \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$.

Example 2.2 (Classical propositional logic)**.** As a running example, we give the institution **CPL** of classical propositional logic. Its signatures are just sets Σ of propositional variables, i.e., $\mathbb{S}ign^{\mathbf{CPL}} = \mathbb{S}et$.$\mathsf{Sen}^{\mathbf{CPL}}(\Sigma)$ is the set of propositional formulas with propositional variables from Σ and connectives for conjunction, disjunction, implication and negation. A signature morphism σ is a mapping between the propositional variables, and sentence translation $\mathsf{Sen}^{\mathbf{CPL}}(\sigma)$ is the extension of σ to all formulas. For example, $A \wedge B \in \mathsf{Sen}^{\mathbf{CPL}}(\{A, B\})$; and for $\sigma : \{A, B\} \to \{C, D\}$, $\sigma : A \mapsto C, B \mapsto D$, we have $\mathsf{Sen}^{\mathbf{CPL}}(\sigma)(A \wedge B) = C \wedge D$. Models of Σ are truth valuations, i.e., mappings from Σ into the standard two-valued Boolean algebra $Bool = \{0, 1\}$ with 0 denoting false and 1 denoting truth. A unique model morphism between Σ-models M and M' exists iff for all $p \in \Sigma$, $M(p) \leqslant M'(p)$. Given $\sigma{:}\Sigma_1 \to \Sigma_2$ and a Σ_2-model $M_2{:}\Sigma_2 \to Bool$, the reduct $M_2|\sigma$ is just the composition $M_2 \circ \sigma$. Reducts of morphisms are then clear. Finally, $M \models^{\mathbf{CPL}}_{\Sigma} \varphi$ holds iff φ evaluates to 1 under the usual extension of M to all formulas. For example, $M : A \mapsto 1, B \mapsto 0$ is an object of $\mathsf{Mod}^{\mathbf{CPL}}(\{A, B\})$, and $M \not\models^{\mathbf{CPL}}_{\{A,B\}} A \wedge B$ because $A \wedge B$ evaluates to $1 \wedge 0 = 0$.

Example 2.3 (Classical first-order logic)**.** In the institution **FOL** of first-order logic, signatures are first-order signatures, consisting of function and predicate symbols with arity. Signature morphisms map symbols such that arity is preserved. Sentences are first-order formulas extending $\mathsf{Sen}^{\mathbf{CPL}}$ with variables, terms, and universal and existential quantification. Sentence translation means replacement of the translated symbols. Models are first-order structures (U, ν) with a universe U and an interpretation function ν assigning functions and relations of appropriate arities to the function and predicate symbols. Nullary predicate symbols are interpreted as an element of *Bool*. Model morphisms are the usual homomorphisms between first-order structures (the need to preserve but not necessarily reflect the holding of predicates). Model reduction means reassembling the model's components according to the signature morphism. Satisfaction is the usual satisfaction of a first-order sentence in a first-order structure. We do not go into the details here because we only need **FOL** to give an example of an institution comorphism below.

Proof-Theoretic extensions of institutions use richer structures than sets of sentences to express the syntax of an institution. They were first explored in [19] where for each signature, there is a category of proofs, with sentences as objects and proofs as morphisms. Here, we introduce preorder-enriched institutions that extend this with reductions between proof terms, modeled by a preorder on morphisms. We write $f \rightsquigarrow g$ to express that f reduces to g for two proofs f, g with the same domain and codomain. In other words, proof categories are small preorder-enriched categories, and we write $\mathbb{O}rdCat$ for the category of preorder-enriched small categories. If $U : \mathbb{O}rdCat \to \mathbb{S}et$ is the functor forgetting morphisms (i.e., $U(A)$ is the set of objects of A), this is formally defined as follows.

Definition 2.4. A *preorder-enriched institution* is a tuple $(\mathbb{S}ign, \mathsf{Pr}, \mathsf{Mod}, \models)$ where $\mathsf{Pr} : \mathbb{S}ign \to \mathbb{O}rdCat$ is a functor such that $(\mathbb{S}ign, U \circ \mathsf{Pr}, \mathsf{Mod}, \models)$ is an institution. In that case, we will use Sen to abbreviate $U \circ \mathsf{Pr}$.

Example 2.5. **CPL** can be turned into a preorder-enriched institution in various ways. Let $\mathbf{CPL}^{ND\lambda}$ denote propositional logic with natural deduction. Then $\mathsf{Pr}^{\mathbf{CPL}}(\Sigma)$ is the category where the objects are propositional formulae over Σ, and morphisms from φ to ψ are natural deduction proof terms (modulo α-congruence) in context $x : \varphi \rhd M : \psi$ as given by the rules in Fig. 2, cf.[6]. The assumption rule gives us $x : \varphi \rhd x : \varphi$ as the identity morphism; and the composition of morphisms $x : \varphi \rhd M : \psi$ and $y : \psi \rhd N : \chi$ is given by substitution: $x : \varphi \rhd N[y := M] : \chi$. The

$$\textbf{(assumption)}\ \frac{}{\Gamma, x : A \rhd x : A}$$

$$(\top\textbf{-I})\ \frac{}{\Gamma \rhd \Delta : \top}$$

$$(\wedge\textbf{-I})\ \frac{\Gamma \rhd M : A \quad \Gamma \rhd N : B}{\Gamma \rhd \langle M, N \rangle : A \wedge B} \qquad (\wedge\textbf{-E1})\ \frac{\Gamma \rhd M : A \wedge B}{\Gamma \rhd \mathsf{fst}(M) : A} \qquad (\wedge\textbf{-E2})\ \frac{\Gamma \rhd M : A \wedge B}{\Gamma \rhd \mathsf{snd}(M) : B}$$

$$(\to\textbf{-I})\ \frac{\Gamma, x : A \rhd M : B}{\Gamma \rhd \lambda x{:}A.M : A \to B} \qquad (\to\textbf{-E})\ \frac{\Gamma \rhd M : A \to B \quad \Gamma \rhd N : A}{\Gamma \rhd MN : B}$$

$$(\bot\textbf{-E})\ \frac{\Gamma \rhd M : \bot}{\Gamma \rhd \nabla_A(M) : A}$$

$$(\vee\textbf{-I1})\ \frac{\Gamma \rhd M : A}{\Gamma \rhd \mathsf{inl}(M) : A \vee B} \qquad (\vee\textbf{-I2})\ \frac{\Gamma \rhd M : B}{\Gamma \rhd \mathsf{inr}(M) : A \vee B}$$

$$(\vee\textbf{-E})\ \frac{\Gamma \rhd M : A \vee B \quad \Gamma, x : A \rhd N : C \quad \Gamma, y : B \rhd P : C}{\Gamma \rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \mid \mathsf{inr}(y) \to P\ :\ C}$$

$$\textbf{(tnd)}\ \frac{}{\Gamma \rhd \mathsf{tnd}_A : A \vee \neg A}$$

Figure 2. Natural deduction proof rules and proof terms for classical propositional logic, introduced in five stages: Void logic, conjunctive logic, conjunctive-implicational logic, intuitionistic ic logic, classical logic

category laws easily follow from general properties of substitutions. The pre-order on these morphisms is given by β-reduction on proof terms as given in Fig. 3. For a signature morphism $\sigma : \Sigma \to \Sigma'$, the functor $\mathsf{Pr}^{\mathbf{CPL}}(\sigma)$ acts on objects of $\mathsf{Pr}^{\mathbf{CPL}}(\Sigma)$ like $\mathsf{Sen}^{\mathbf{CPL}}(\sigma)$; similarly, $\mathsf{Pr}^{\mathbf{CPL}}(\sigma)$ acts on morphisms of $\mathsf{Pr}^{\mathbf{CPL}}(\Sigma)$, i.e., on natural deduction proofs, by replacing all occurrences of a symbol of Σ with its image under σ.

We define a functor $\overline{\cdot} : \mathbb{O}rdCat \to \mathbb{CAT}$ by quotienting out the preorder; i.e., given a preorder-enriched category A, $\overline{A}$ is the quotient of A by the equivalence generated by the preorder on hom-sets. Similarly, $thin(\cdot) : \mathbb{O}rdCat \to \mathbb{CAT}$ is a functor that quotients all non-empty hom-sets (i.e. sets of 1-cells) to singletons.

In any institution, we have the usual notion of *semantic consequence*: For a set Φ of Σ-sentences, let $M \models_\Sigma \Phi$ denote $M \models_\Sigma \varphi$ for every $\varphi \in \Phi$. Then we say that that a Σ-sentence ψ is a consequence of Φ, and write $\Phi \models_\Sigma \psi$, iff $M \models_\Sigma \Phi$ implies $M \models_\Sigma \psi$ for every Σ-model M.

In a preorder-enriched institution, we can also define an *entailment* relation $\vdash_\Sigma$ between Σ-sentences as follows: $\varphi \vdash_\Sigma \psi$ iff there exists a morphism $\varphi \to \psi$ in $\mathsf{Pr}(\Sigma)$. A preorder-enriched institution is *sound* if $\phi \vdash_\Sigma \psi$ implies $\{\phi\} \models_\Sigma \psi$, and *complete* if the converse implication holds.

$$
\begin{array}{lll}
M : \top & \rightsquigarrow_\beta & \Delta : \top \\
\mathsf{fst}(\langle M, N\rangle) & \rightsquigarrow_\beta & M \\
\mathsf{snd}(\langle M, N\rangle) & \rightsquigarrow_\beta & N \\
(\lambda x{:}A.M)N & \rightsquigarrow_\beta & M[x := N] \\
\mathsf{case}\ \mathsf{inl}(M)\ \mathsf{of}\ \mathsf{inl}(x) \to N \mid \mathsf{inr}(y) \to P & \rightsquigarrow_\beta & N[x := M] \\
\mathsf{case}\ \mathsf{inr}(M)\ \mathsf{of}\ \mathsf{inl}(x) \to N \mid \mathsf{inl}(y) \to P & \rightsquigarrow_\beta & P[x := M]
\end{array}
$$

Figure 3. β-reduction rules for proof terms

In an institution $\mathcal{I}$, a *theory* is a pair $T = \langle \Sigma, \Gamma\rangle$, where $\Sigma \in |\mathbb{S}ign|$ and $\Gamma \subseteq \mathsf{Sen}(\Sigma)$. A *theory morphism* $\sigma{:}\langle\Sigma, \Gamma\rangle \to \langle\Sigma', \Gamma'\rangle$ is a signature morphism $\sigma{:}\Sigma \to \Sigma'$ for which $\Gamma' \models_{\Sigma'} \sigma(\Gamma)$, that is, σ maps axioms to consequences.

This defines a category **Th** of theories, and it is easy to extend Sen (or Pr) and Mod to **Th** by putting $\mathsf{Sen}(\langle\Sigma, \Gamma\rangle) = \mathsf{Sen}(\Sigma)$ and letting $\mathsf{Mod}(\langle\Sigma, \Gamma\rangle)$ be the full subcategory of $\mathsf{Mod}(\Sigma)$ induced by the class of those models M satisfying Γ.

Relationships between institutions are captured by several variants of institution morphisms and comorphisms. Here we use the following definition.

Definition 2.6. Given preorder-enriched institutions $\mathcal{I}$ and $\mathcal{J}$, a *comorphism* between them is a tuple (Φ, α, β) consisting of

- a functor Φ: $\mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}ign^{\mathcal{J}}$,
- a natural transformation α: $\mathsf{Pr}^{\mathcal{I}} \to \mathsf{Pr}^{\mathcal{J}} \circ \Phi$,
- a natural transformation β: $\mathsf{Mod}^{\mathcal{J}} \circ \Phi^{op} \to \mathsf{Mod}^{\mathcal{I}}$

such that the following satisfaction condition holds for all $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, $M' \in |\mathsf{Mod}^{\mathcal{J}}$ $(\Phi(\Sigma))|$ and $\varphi \in |\mathsf{Sen}^{\mathcal{I}}(\Sigma)|$:

$$M' \models^{\mathcal{J}}_{\Phi(\Sigma)} \alpha_\Sigma(\varphi) \ \text{ iff } \ \beta_\Sigma(M') \models^{\mathcal{I}}_\Sigma \varphi.$$

Example 2.7. To understand the intuition behind this definition, assume that the institution **FOL** of first-order logic from Example 2.3 is extended to a preorder-enriched institution $\mathbf{FOL}^{ND\lambda}$ by extending the natural deduction calculus used in Example 2.5 with rules for the quantifiers. Then the natural inclusion of propositional into first-order logic is formalized as a comorphism (Φ, α, β) from $\mathbf{CPL}^{ND\lambda}$ to $\mathbf{FOL}^{ND\lambda}$.

For every $\mathbf{CPL}^{ND\lambda}$-signature Σ, $\Phi(\Sigma)$ is the first-order signature with Σ as the set of nullary predicate symbols and with no other function or predicate symbols. The sentence and proof translation α_Σ is an inclusion functor from $\mathsf{Pr}^{\mathbf{CPL}}(\Sigma)$ to $\mathsf{Pr}^{\mathbf{FOL}}(\Phi(\Sigma))$ because every propositional formula and every propositional proof over Σ is also a first-order formula or proof over $\Phi(\Sigma)$. For a model $(U, \nu) \in |\mathsf{Mod}^{\mathbf{FOL}}(\Phi(\Sigma))|$, ν is simply a valuation $\Sigma \to Bool$; and thus we can put $\beta_\Sigma(U, \nu) = \nu$. A model morphism in $\mathsf{Mod}^{\mathbf{FOL}}(\Phi(\Sigma))$ needs to preserve holding of nullary predicates and hence induces a model morphism in $\mathsf{Mod}^{\mathbf{CPL}}(\Sigma)$. The satisfaction condition, intuitively, requires that truth and consequence are preserved under a comorphism. It is easy to verify because $\alpha_\Sigma(\varphi) = \varphi$ and $\beta_\Sigma(U, \nu) = \nu$ for a model $M' = (U, \nu)$ and because $\models^{\mathbf{FOL}}$ agrees with $\models^{\mathbf{CPL}}$ on propositional formulas.

Example 2.8. The well-known Curry-Howard isomorphism can be formulated as an institution isomorphism between $\mathbf{CPL}^{ND\lambda}$ and $\mathbf{CPL}^{ND}$, where $\mathbf{CPL}^{ND}$ is the standard formulation of natural deduction, with proof trees as morphisms in proof categories, and proof tree reduction as the preorder.

Together with the obvious composition and identities, this defines a category[1] $\mathbb{C}oIns$ of preorder-enriched institutions and comorphisms.

3 Categorical Logic

Lambek and Scott[26] study categorical logic by introducing *deductive systems* as directed graphs with a composition structure, and later impose the usual axioms for (cartesian, cartesian closed, bicartesian closed) categories on these. Objects in categories serve as 'types' for morphisms, hence the 'propositions as types' paradigm becomes 'propositions as objects'.

We will use a formal meta-language to formalize categorical logic, namely an institution $\mathcal{DFOL}$ that extends many-sorted first-order logic with dependent sorts. $\mathcal{DFOL}$ is formally introduced in [32], and we will only sketch its definition here.

The idea of $\mathcal{DFOL}$ is to add a restricted version of dependent sorts in a way that preserves much of the meta-theory of first-order logic. A $\mathcal{DFOL}$-signature consists of a sequence of partial signatures, called levels. Each level is a sequence of typed symbol declarations. A declaration is of the form $s : \Pi x : S_1. \ldots .\Pi x_n : S_n.T$; here every S_i is an instance of a dependent sort, i.e., a sort symbol followed by argument terms according to its type, and every S_i may contain the variables $x_1, \ldots, x_{i-1}$; and T is one the following three depending on whether s is a sort, function, or predicate symbol, respectively: *sort*, an instance of a dependent sort possibly containing the variables $x_1, \ldots, x_n$, or $Form$.[2]

These declarations are subject to certain conditions. On level 0, only base sorts, i.e., sort symbols that do not take arguments, may be declared. Thus, level 0 is exactly a signature of many-sorted first-order logic. And on level $n+1$, only sort symbols may be declared whose argument sorts contain only symbols from at most level n. Furthermore, a function symbol declared on level $n+1$ must have a return sort that is also declared on level $n+1$.

For example, if level 0 declares a sort $S : sort$, level 1 may declare a sort $S' : \Pi x : S.sort$ that depends on arguments of the level 0 sort S, and a function symbol $f : \Pi x : S.S'(x)$; a function symbol declaration $f' : \Pi x : S.\Pi y : S'(x).S$ is forbidden because f' would return a term of the level 0 sort S although there is an argument sort $S'(x)$ of level 1. As usual when dealing with dependent types, we write $\Pi x : S.S'$ as $S \to S'$ if x does not occur freely in S'. We also write curried $\to$ as $\times$, i.e., $S \to S' \to S''$ as $S \times S' \to S''$.

The atomic formulas over a $\mathcal{DFOL}$-signature Σ are given by an application of a predicate symbol to terms according to its type or by an application of equality

[1] Of course, this category lives in a higher set-theoretic universe. We omit the foundational issues here.

[2] For simplicity, we omit the $\mathcal{DFOL}$-symbol $Univ$ here.

to terms of the same sort. Then the sentences are built up in the same way as for many-sorted first-order logic. In particular, the sentences use the symbols $\wedge$, $\Longrightarrow$, $==$ and $\forall$. The quantification has the additional subtlety that in formulas such as $\forall x : S.\forall y : S(x).\varphi$, the sorts of the variables may be dependent and may contain other variables. We define the level of a formula ϕ to be the highest level of a symbol that occurs in ϕ.

The models of a $\mathcal{DFOL}$-signature Σ are constructed level-wise. A model for the declarations at level 0 is a usual model of many-sorted first-order logic. To extend a model for level n to a model for level $n+1$, every sort symbol of level $n+1$ is interpreted as an assignment of universes to the possible argument tuples according to its type. Then every function and predicate symbol of level 1 can be interpreted as a function or relation according to its type. Then satisfaction is defined in essentially the same way as for many-sorted first-order logic.

To formalize categories, we will use the following $\mathcal{DFOL}$-signature:

$Ob : sort.$
$Des : Ob.$
$Mor : Ob \times Ob \rightarrow sort.$
$id : \Pi A : Ob.Mor(A, A).$
$; : \Pi A, B, C : Ob.\, Mor(A, B) \times Mor(B, C) \rightarrow Mor(A, C).$
$\rightsquigarrow : \Pi A, B : Ob.\, Mor(A, B) \times Mor(A, B) \rightarrow Form.$

Here, *Ob* is a base sort, and *Des* is a constant of sort *Ob*. They are the only declaration on level 0. All remaining declarations are on level 1. Firstly, *Mor* is a dependent sort, its type means that for all terms A, B of sort *Ob*, $Mor(A, B)$ is a sort. *id* is a function symbol that takes an argument of type *Ob* and returns a term of a sort that depends on the argument. Similarly ; takes three arguments of sort *Ob*, say A, B and C, and then two arguments of the sorts $Mor(A, B)$ and $Mor(B, C)$ and returns a term of sort $Mor(A, C)$. $\rightsquigarrow$ is a predicate symbol: It takes two arguments of sort *Ob*, say A and B, and two arguments of sort $Mor(A, B)$ and returns an atomic formula.

The intended semantics is that *Ob* is interpreted as the set of objects of a small category (via the Curry-Howard interpretation: formulas of a categorical logic), $Mor(A, B)$ as the set of morphisms (proofs) from A to B, $id(A)$ as the identity (self-proof) of A, $;(A, B, C, f, g)$ as the composition of $f : Mor(A, B)$ and $g : Mor(B, C)$ (the proof of C from A obtained from applying cut with B), and $\rightsquigarrow(A, B, f, g)$ as a relation on morphisms that expresses that $f \rightsquigarrow g$ (the reducibility of f to g). *Des* singles out a certain object (the minimal designated truth value). For simplicity, we will write $f; g$ instead of $;(A, B, C, f, g)$ and $f \rightsquigarrow g$ instead of $\rightsquigarrow(A, B, f, g)$. Similarly, we drop the arguments of sort *Ob* from other function symbols, like π_1 below, if they are clear from the context.

To achieve the intended semantics, we define the $\mathcal{DFOL}$-theory *CatLog* by adding to the above signature the axioms

$\forall A, B, C, D\colon Ob\ \ \forall f, f', f'' : Mor(A, B)\ \ \forall g, g'\colon Mor(B, C)\ \ \forall h\colon Mor(C, D)$
$id(A); f == f$
$h;\, id(D) == h$
$(f;g);h == f;(g;h)$
$f \rightsquigarrow f$

$f \rightsquigarrow f' \wedge f' \rightsquigarrow f'' \Longrightarrow f \rightsquigarrow f''$
$f \rightsquigarrow f' \wedge g \rightsquigarrow g' \Longrightarrow f;\ g \rightsquigarrow f';\ g'$

Then we have indeed that the model category $Mod^{\mathcal{DFOL}}(CatLog)$ is essentially $\mathbb{O}rdCat$. Precisely, a model of *CatLog* is a preorder-enriched category with one distinguished object interpreting *Des*.

We can axiomatize other enriched categories by adding function symbols to *CatLog*. We permit the addition of two kinds of declarations. Firstly, function symbols on level 0; these simply take n terms of sort *Ob* and return a term of sort *Ob*. Such function symbols represent propositional connectives. And secondly, function symbols on level 1; these must take terms of sort *Ob* or an instance of *Mor* and return a term of a sort that is an instance of *Mor*. Such function symbols represent proof rules. Furthermore, we permit certain axioms that represent equalities and rewrites between proofs. The axioms must be in Horn form in order to apply the free model theorem for $\mathcal{DFOL}$[32]. Formally, we define this as follows.

Definition 3.1. A *categorical logic* $\mathcal{L}$ is an extension of *CatLog* with function symbols and axioms such that

- axioms are in Horn form
- if an axiom is of level 1, its head must be an equality or rewrite between morphisms
- there are congruence axioms for $\rightsquigarrow$, i.e., for every new function symbol of the form

  $$c : \Pi \vec{A} : Ob.\ Mor(S_1, T_1) \to \ldots \to Mor(S_n, T_n) \to Mor(S, T)$$

  where $\vec{A}$ abbreviates $A_1, \ldots, A_m$ and S_i, T_i, S, T are terms of the sort *Ob* with free variables $\vec{A} : Ob$, there is an axiom

  $\forall \vec{A}: Ob\ \ \forall f_1, f_1': Mor(S_1, T_1) \ldots \forall f_n, f_n': Mor(S_n, T_n)$
  $(f_1 \rightsquigarrow f_1' \wedge \ldots \wedge f_n \rightsquigarrow f_n') \Longrightarrow c(\vec{A}, f_1, \ldots, f_n) \rightsquigarrow c(\vec{A}, f_1', \ldots, f_n')$

The category of categorical logics, denoted by *CatLog*, has such theories $\mathcal{L}$ as objects and $\mathcal{DFOL}$-theory morphisms that are the identity for the symbols of *CatLog* as morphisms. (More generally, we could use the slice category of $\mathcal{DFOL}$-theories with domain *CatLog*.)

Then for any categorical logic $\mathcal{L}$, an element of $\mathsf{Mod}^{\mathcal{DFOL}}(\mathcal{L})$ consists of an element of $\mathbb{O}rdCat$ along with interpretations for the symbol *Des* and the added function symbols. $\mathsf{Mod}^{\mathcal{DFOL}}(\mathcal{L})$-morphisms are functors in $\mathbb{O}rdCat$ that preserve the interpretation of *Des* and that commute with the added function symbols. We call this category $\mathbb{C}(\mathcal{L})$, its elements $\mathcal{L}$-categories, and its morphisms $\mathcal{L}$-functors. In an $\mathcal{L}$-category A, the preorders on all hom-sets are denoted by $\rightsquigarrow^A$; and the interpretation of a function symbol c is denoted by c^A. Then the usual notion of (small) enriched categories becomes a special case of $\mathcal{L}$ categories.

We have the following result:

Proposition 3.2. *For every categorical logic $\mathcal{L}$ and every set X of variables of object or morphism sort, there is a free model $T_{\mathcal{L}}(X) \in |\mathbb{C}(\mathcal{L})|$, i.e., for every $A \in |\mathbb{C}(\mathcal{L})|$ and every mapping m that maps variables in X to objects or morphisms* *of A according to their sort, there is a unique extension* $\overline{m}$ *of* m *to an* $\mathcal{L}$*-functor* $T_{\mathcal{L}}(X) \to A$.

Proof: This is a special case of the result proven in [32]. Therefore, we only give the idea of the proof. In the first step, only the sort *Ob*, the variables of sort *Ob*, the function symbols on level 0, and all axioms of level 0 are considered. These induce a free term model in the usual way of first-order logic. Then in a second step, all elements of the universe of this free model are added to the signature as constants; and then another term model construction can be used to obtain the desired model.

For the correctness of the construction, it is crucial that there are no function symbols of level 1 that return a term of sort *Ob*, and no axioms of level 1 that might imply an equality between terms of the *Ob*; this ensures that the model found in the first step is not influenced by the second step. □

The objects and morphisms of $T_{\mathcal{L}}(X)$ are equivalence classes of terms. For simplicity, we use every term as an abbreviation for its equivalence class in $T_{\mathcal{L}}(X)$.

Example 3.3 (Cartesian Categories). The categorical logic *Cartesian* arises by extending *CatLog* with declarations

$\top : Ob$
$* : Ob \times Ob \to Ob$
$! : \Pi A : Ob.\ Mor(A, \top).$
$\langle \_, \_ \rangle : \Pi A, B, C : Ob.\ Mor(A, B) \times Mor(A, C) \to Mor(A, B * C)$
$\pi_1 : \Pi A, B : Ob.\ Mor(A{*}B, A)$
$\pi_2 : \Pi A, B : Ob.\ Mor(A{*}B, B)$

and axioms

$f : A \to \top \rightsquigarrow\ !$
$\langle f;\, \pi_1, f;\, \pi_2 \rangle \rightsquigarrow f$
$\langle f, g \rangle;\, \pi_1 \rightsquigarrow f$
$\langle f, g \rangle;\, \pi_2 \rightsquigarrow g$
$\langle f;g, f;h \rangle \rightsquigarrow f;\, \langle g, h \rangle$

making $\top$ a terminal object and $*$ a binary product with projections π_1 and π_2. Logically, this corresponds to adding "true" ($\top$) and conjunction ($*$). The first three rewrite axioms axiomatize the elimination of redundant proof steps, and the last rewrite axiom is a step in the cut (i.e., composition ;) elimination.

Note that we have some freedom in specifying the axioms. For example, we can either add the axiom $\langle f, g \rangle;\, \pi_1 == f$ (omitting the universal quantifiers), which identifies the two proof terms and hence, regards their difference as merely bureaucratic and caused by the needs of syntactic representation. Or we can specify the rewrite rule $\langle f, g \rangle;\, \pi_1 \rightsquigarrow f$, which keeps $\langle f, g \rangle;\, \pi_1$ and f distinct.

Extensions $\mathcal{L}$ of *Cartesian* are called *cartesian*. If $\mathcal{L}$ also has the axiom $Des == \top$, $\mathcal{L}$ is called $\top$*-cartesian*. We will give further examples in Sect. 5.

4 The Institutional Curry-Howard-Tait Construction

Now we construct a preorder-enriched institution out of a categorical logic, thereby following the Curry-Howard-Tait isomorphism paradigm. First, given a categorical logic $\mathcal{L}$, we define two theory extensions: Let $\overline{\mathcal{L}}$ be the extension of $\mathcal{L}$ with the axiom

$$\forall A, B{:}Ob \forall f, g{:}Mor(A, B)\ (f \rightsquigarrow g \Longrightarrow f == g),$$

which, semantically, quotients out the preorder in the $\mathcal{L}$-categories, and let $\overline{\mathcal{L}}$ be the extension of $\overline{\mathcal{L}}$ with the axiom

$$\forall A, B{:}Ob\ \ \forall f, g{:}Mor(A, B)\ f == g,$$

which, semantically, quotients the $\mathcal{L}$-categories to preorders. Let $\overline{\mathbb{C}(\mathcal{L})} = \mathbb{C}(\overline{\mathcal{L}})$ and $\mathbb{C}_{thin}(\mathcal{L}) = \mathbb{C}(\mathcal{L}_{thin})$.

Definition 4.1. Given a categorical logic $\mathcal{L}$, the preorder-enriched institution $\mathcal{I}(\mathcal{L}) = (\mathbb{S}ign, \mathsf{Pr}, \mathsf{Mod}, \models)$ is defined by:

- $\mathbb{S}ign$ is the category of sets (seen as sets of propositional variables).
- Pr is the universal functor of Prop. 3.2. Explicitly, $\mathsf{Pr}(\Sigma) = T_{\mathcal{L}}(\Sigma)$ for a signature Σ, and for a signature morphism (that is, a function) $\sigma{:}\Sigma \to \Sigma'$, $\mathsf{Pr}(\sigma)$ is the unique extension of σ to an $\mathcal{L}$-functor from $T_{\mathcal{L}}(\Sigma)$ to $T_{\mathcal{L}}(\Sigma')$.
- Mod is the lax comma category functor $(\mathsf{Pr}(\cdot) \downarrow \overline{\mathbb{C}(\mathcal{L})})$.

  Explicitly, Mod assigns to a signature Σ a category $\mathsf{Mod}(\Sigma)$ such that

  - objects are pairs (A, m) where $A \in |\overline{\mathbb{C}(\mathcal{L})}|$ and m is a valuation $m{:}\ \Sigma \to |A|$ of the propositional variables to A-objects,[3]
  - model morphisms from (A, m) to (A', m') are pairs (F, μ) where $F{:}\ A \to A'$ is an $\mathcal{L}$-functor and μ is a family of A'-morphisms over $p \in \Sigma$ such that $\mu_p{:}F(m(p)) \to m'(p)$,

  and for a signature morphism $\sigma{:}\Sigma \to \Sigma'$, the model reduct functor $\mathsf{Mod}(\sigma)$ is given by composition: $\mathsf{Mod}(\sigma)(A, m) = (A, m \circ \sigma)$ for models and $\mathsf{Mod}(\sigma)(F, \mu) = (F, \mu \circ \sigma)$ for model morphisms.
- Satisfaction is defined by: $(A, m) \models_{\Sigma} \varphi$ iff $A(Des^A, \overline{m}(\varphi))$ is inhabited.

For example, intuitionistic and classical propositional logic are both captured by this construction. A detailed discussion of examples can be found in Sect. 5.

Proposition 4.2 (Institutionality)**.** For any categorical logic $\mathcal{L}$, $\mathcal{I}(\mathcal{L})$ is a preorder-enriched institution.

Proof: The satisfaction condition is shown as follows. Let $\sigma{:}\ \Sigma \to \Sigma'$ be a morphism, let φ be a Σ-sentence, and let (A', m') be a Σ'-model. Then $(A', m')|_{\sigma} \models_{\Sigma} \varphi$ iff $A'(Des^{A'}, \overline{m' \circ \sigma}(\varphi)) \neq \emptyset$ iff $A'(Des^{A'}, \overline{m'}(\sigma(\varphi))) \neq \emptyset$ iff $(A', m') \models_{\Sigma'} \sigma(\varphi)$. □

For a signature Σ, the Curry-Howard-Tait correspondence then takes the following shape:

[3]Remember that such a valuation uniquely determines an $\mathcal{L}$-functor $\overline{m}$ from $\mathsf{Pr}(\Sigma)$ to A.

Propositions as types/objects Sentences are the objects of $\mathsf{Pr}(\Sigma)$, i.e., sentences are $\mathcal{L}$-terms of sort Ob with propositional variables from Σ.

Proofs as terms Proofs are the morphisms of $\mathsf{Pr}(\Sigma)$. That is, a Σ-proof between sentences φ and ψ is simply an equivalence class of $\mathcal{L}$-terms of sort $Mor(\varphi,\psi)$. If the only equations in $\mathcal{L}$ are those of $CatLog$, this means that Σ-proofs are strings of composable composition-free proof terms.

Proof reduction as morphism ordering The reducibility of proofs is given by the $\rightsquigarrow$ predicate in $T_{\mathcal{L}}(\Sigma)$, which is a preorder on morphisms (which is preserved under composition).

Categorical models A Σ-model is determined by a category A in $|\overline{\mathbb{C}(\mathcal{L})}|$ and a valuation of the propositional variables into A. Proof reduction is modeled by the interpretation of $\rightsquigarrow$.

Satisfaction via designated truth values For a Σ-model (A,m), the preorder $thin(A)$ gives the truth values of A. Then the set of designated truth values is the upper set of Des^A in $thin(A)$. (Defining more complex sets of designated truth values is possible by making Des a predicate on objects instead of a constant, as in [2].)

In the remainder of this section, we derive some crucial properties of this construction.

Proposition 4.3 (Soundness). *For every categorical logic $\mathcal{L}$, $\mathcal{I}(\mathcal{L})$ is sound.*

Proof: Let $\phi\vdash_\Sigma\psi$, i.e. we have a morphism $p:\phi\to\psi$ in $\mathsf{Pr}(\Sigma)$. Let (A,m) be a Σ-model such that $(A,m)\models\phi$, i.e. $A(Des^A,\bar{m}(\phi))\neq\emptyset$. Postcomposition with the morphism $\bar{m}(p):\bar{m}(\phi)\to\bar{m}(\psi)$ yields $A(Des^A,\bar{m}(\psi))\neq\emptyset$, i.e. $(A,m)\models\psi$. □

For a categorical logic $\mathcal{L}$, we put $\vdash_\Sigma\psi$ iff $\mathsf{Pr}(\Sigma)(Des,\psi)$ is non-empty. It is trivial to note that $\mathcal{I}(\mathcal{L})$ is *weakly sound*, i.e. $\vdash_\Sigma\psi$ implies $\emptyset\models_\Sigma\psi$. We say that $\mathcal{I}(\mathcal{L})$ is *weakly complete* if the converse implication holds.

When $\mathcal{L}$ is cartesian, then we have a binary connective $*$ (conjunction) on sentences of $\mathcal{I}(\mathcal{L})$ arising from binary product $\times$; moreover, the terminal object $\top$ induces a sentence $\top$ (truth). We can then extend the entailment relation $\vdash$ to sets of hypotheses by putting $\Phi\vdash_\Sigma\psi$ if there exist $\varphi_1,\ldots,\varphi_n\in\Phi$ such that

$$\varphi_1*\cdots*\varphi_n\vdash_\Sigma\psi$$

where we put $\varphi_1*\ldots*\varphi_n=\top$ if $n=0$. In this notation, $\phi\vdash_\Sigma\psi$ is equivalent to $\{\phi\}\vdash_\Sigma\psi$, and $\vdash_\Sigma\psi$ is equivalent to $\emptyset\vdash_\Sigma\psi$. In this setting, we say that $\mathcal{I}(\mathcal{L})$ is *strongly sound* if $\Phi\vdash_\Sigma\psi$ implies $\Phi\models_\Sigma\psi$, that $\mathcal{I}(\mathcal{L})$ is *strongly complete* if the converse implication holds.

Proposition 4.4 (Strong soundness). *For any cartesian categorical logic $\mathcal{L}$, $\mathcal{I}(\mathcal{L})$ is strongly sound.*

Proof: By Proposition 4.3, it suffices to show that $\Phi\models_\Sigma\varphi_1*\ldots\varphi_n$ for all $\varphi_1,\ldots\varphi_n\in\Phi$. If $(A,m)\models\Phi$ for a model (A,m), then we can pick $q_i\in A(Des^A,\bar{m}(\phi_i))$ for $i=1,\ldots,n$. Then $\langle q_1,\ldots,q_n\rangle\in A(Des^A,\bar{m}(\phi_1)*\ldots*\bar{m}(\phi_n))\cong A(Des^A,\bar{m}(\phi_1*\ldots*\phi_n))$ and hence $(A,m)\models\phi_1*\ldots*\phi_n$.

It is important to recognize that, unlike the categories A in Σ-models (A,m), the proof categories $\mathsf{Pr}(\Sigma)$ are generally *not* elements of $\overline{\mathbb{C}(\mathcal{L})}$. This avoids the problem stated in [26] that for classical propositional logic, the categorical semantics of proofs collapses: All classical bicartesian closed categories are partial orders (and thus boolean algebras). If one does not wish to distinguish between rewritable proofs, one can collapse the proof categories by using $\overline{\mathcal{I}(\mathcal{L})} := \mathcal{I}(\overline{\mathcal{L}})$. Proof irrelevance is obtained by using $\mathcal{I}_{thin}(\mathcal{L}) := \mathcal{I}(\mathcal{L}_{thin})$.

Definition 4.5 (Deduction Theorem). We say that a categorical logic $\mathcal{L}$ satisfies the *weak deduction theorem* if for every Σ-proof term $p : Mor(Des, \psi)$ in one free variable $x{:}Mor(Des, \chi)$, there exists a closed Σ-proof term $\kappa x.p$ of type $Mor(\chi, \psi)$. If $\mathcal{L}$ is cartesian, then $\mathcal{L}$ satisfies the *deduction theorem* if for every Σ-proof term $p : Mor(\varphi, \psi)$ in free variables χ, $x{:}Mor(Des, \chi)$, there exists a Σ-proof term $\kappa x.p$ of type $Mor(\varphi{*}\chi, \psi)$ in free variables X.

Theorem 4.6 (Completeness). *For every categorical logic $\mathcal{L}$, $\mathcal{I}(\mathcal{L})$ is weakly complete. If $\mathcal{L}$ satisfies the weak deduction theorem, then $\mathcal{L}$ is complete. If $\mathcal{L}$ is $\top$-cartesian and satisfies the deduction theorem, then $\mathcal{I}(\mathcal{L})$ is strongly complete.*

Proof: We prove only the third claim; the first two follow by analogous, but simpler arguments. Assume $\Phi \models_\Sigma \psi$. Let $F \in |\mathsf{Mod}^{\mathcal{DFOL}}(\mathcal{L})|$ be the free $\mathcal{L}$-category existing by Prop. 3.2 over the following sorted variables: variables $v : Ob$ for every $v \in \Sigma$ and variables $x_\varphi : Mor(\top, \varphi)$ for all $\varphi \in \Phi$. Then $F \models_\Sigma \varphi$ for all $\varphi \in \Phi$. Hence, by the assumption of the theorem, we have $F \models_\Sigma \psi$. Then, since F is a free term model differing from $\mathsf{Pr}(\Sigma)$ only by having more variables, there must be a $\mathsf{Pr}(\Sigma)$-term $p(x_\varphi, \ldots) : Mor(\top, \psi)$ in free variables x_φ. Clearly, p can only refer to finitely many x_φ. Thus repeated application of the deduction theorem yields a closed $\mathsf{Pr}(\Sigma)$-term of type $Mor(\top{*}\varphi_1{*}\ldots{*}\varphi_n, \psi)$, and precomposition with the isomorphism $\varphi_1{*}\ldots{*}\varphi_n \cong \top{*}\varphi_1{*}\ldots{*}\varphi_n$ yields $\phi_1{*}\ldots{*}\phi_n{\vdash}\psi$ and hence $\Phi{\vdash}\psi$. □

It should be noted that the fact that this general completeness theorem has a rather easy proof is caused by its restricted applicability, mainly to propositional logics (there is no handling of variables and quantification). Even for most modal propositional logics, the essential assumption for the second result, namely the deduction theorem, fails. However, there is an easy criterion for obtaining the deduction theorem:

Proposition 4.7. *The deduction theorem holds in a $\top$-cartesian categorical logic provided that newly introduced function symbols that return morphisms do not take morphisms as arguments (that is, there are* no new logical rules).

Proof: See Proposition I.2.1 of [26]. The addition of operations adhering to the above restriction does not destroy the induction proof given there.

Example 4.8. Let $Cartesian_{top}$ be the $\top$-cartesian categorical logic arising by only adding $Des == \top$ to $Cartesian$. The institution $\mathcal{I}(Cartesian_{top})$ is described as follows. Signatures are sets of propositional variables. Sentences are the corresponding fragment of propositional logic. A model consists of a category together with an interpretation of the propositional variables as objects in this category. The conjunction is interpreted by a product, and truth by a terminal element. Evaluation of sentences is just term evaluation in the category. The designated truth values are $\top$ and everything provable from it.

If $p{:}\ C \to A$ and $q{:}C \to B$ are proofs in a model (A, m), they can be combined to $\langle p, q\rangle^A{:}C \to A{*}^A B$, and $\langle p, q\rangle^A;\ \pi_1^A$ is another proof from C to A. The rewriting structure gives us $\langle p, q\rangle^A;\ \pi_1^A{\rightsquigarrow}^A p$. In the institution $\overline{\mathcal{I}(Cartesian_{top})}$, these two proofs are identified.

We arrive at cartesian categories[26] if we quotient out rewrites, i.e., use $\mathcal{I}_{thin}$ $(Cartesian_{top})$.

We can define cut elimination in the context of categorical logic[15]: The cut rule corresponds to the composition of morphisms. Then *cut elimination* means that the composition operation can be eliminated from proof terms.

Defintion 4.9 (Completeness)**.** categorical logic $\mathcal{L}$ *admits cut* if for all signatures Σ and all objects $A, B \in |T_{\mathcal{L}}(\Sigma)|$, for every Σ-proof term $p : Mor(A, B)$ there is a Σ-proof term $p' : Mor(A, B)$ such that p' does not contain the function symbol. $\mathcal{L}$ has cut elimination if, in addition, $p {\leadsto} p'$ holds in $T_{\mathcal{L}}(\Sigma)$.

This analogy is not perfect, however, because not in all logics, the more complex cut rules with multiple formulas on each side of the turnstile can be derived from this simple version of cut. A solution might be to use polycategories[41,3], but this is beyond the scope of this work.

Both cut admissibility and cut elimination need to be established independently for every categorical logic: Minor changes in the specification can destroy these properties, or require redoing large portions of their proofs. While some of the above examples have cut admissibility, additional rewrites are necessary to establish cut elimination. These additional rewrites correspond to the various cases and subcases of the induction step of a constructive cut elimination proof.

Proposition 4.10 (Completeness)**.** *Cartesian* has cut elimination.

Proof: The proof proceeds by nested term inductions on f and g in $f; g$. This result can also be found in [15]. □

Exactness of institutions is a property important for modular specifications and proofs[35]:

Definition 4.11. An institution is said to be *semi-exact*, if any pushout

$$\begin{array}{ccc} \Sigma & \longrightarrow & \Sigma_1 \\ \downarrow & & \downarrow \\ \Sigma_2 & \longrightarrow & \Sigma_R \end{array}$$

in $\mathbb{S}ign$ is mapped by Mod to a pullback

$$\begin{array}{ccc} \mathsf{Mod}(\Sigma) & \longleftarrow & \mathsf{Mod}(\Sigma_1) \\ \uparrow & & \uparrow \\ \mathsf{Mod}(\Sigma_2) & \longleftarrow & \mathsf{Mod}(\Sigma_R) \end{array}$$

of categories. Explicitly, this means that any pair $(M_1, M_2) \in \mathsf{Mod}(\Sigma_1) \times \mathsf{Mod}(\Sigma_2)$ that is *compatible* in the sense that M_1 and M_2 reduce to the same Σ-model can be *amalgamated* to a unique Σ_R-model M (i.e., there exists a unique $M \in \mathsf{Mod}(\Sigma_R)$ that reduces to M_1 and M_2, respectively), and similarly for model morphisms.

Proposition 4.12. *For any categorical logic $\mathcal{L}$, $\mathcal{I}(\mathcal{L})$ is semi-exact.*

Proof : This follows immediately because the model reduct functor $\mathsf{Mod}(\sigma)$ is defined by composition of valuations with signature morphisms. □

Proposition 4.13. An institution is said to be (*weakly*) *liberal*, if the reduct functor of each *theory morphism* σ: $(\Sigma_1, \Psi_1) \to (\Sigma_2, \Psi_2)$ has a (weak) left adjoint. A functor F is a weak left adjoint to U via unit η: $Id \to UF$, if any morphism f: $X \to U\,A$ factors (not necessarily uniquely) as $U\,g \circ \eta_X$ for some g: $F\,X \to A$.

$\mathcal{I}(IProp)$ and $\mathcal{I}(Prop)$ are not liberal: the theory $(\{A,B\},\{A+B\})$ (viewed as extension of the empty theory) has no free model. However, we have

Proposition 4.14. For any categorical logic $\mathcal{L}$ with truth, $\mathcal{I}(\mathcal{L})$ is weakly liberal.

Proof : Given a Σ_1-model (A,m), the (Σ_2,Ψ_2)-*diagram* of (A,m) is obtained as follows. Add all objects of A as propositional variables to Σ_2, arriving at the signature $\Sigma_2(A)$; (A,m) is easily extended to a model of that signature. Add all $\Sigma_2(A)$-sentences holding in (A,m) to Ψ_2, obtaining a theory $\Psi_2(A)$. By Prop. 3.2, there is a free $\mathcal{L}(\Sigma_2(A) \cup \{x_\varphi : Mor(\top,\varphi) \,|\, \varphi \in \Psi_2(A)\})$-model. It canonically is a Σ_2-model, which is a weakly free extension of (A,m). □

Proposition 4.15. *The construction $\mathcal{I}\,(-)$ is functorial, i.e., it can be extended to a functor $\mathcal{I}(-)$: $\mathbb{C}atLog \to \mathbb{C}oIns$.*

Proof : Given two categorical logics $\mathcal{L}_1$ and $\mathcal{L}_2$ and a $\mathcal{DFOL}$-theory morphism σ:$\mathcal{L}_1 \to \mathcal{L}_2$, we can construct a preorder-enriched institution comorphism (Φ,α,β): $\mathcal{I}(\mathcal{L}_1) \to \mathcal{I}(\mathcal{L}_2)$ as follows:

- Φ is the identity functor in the category $\mathbb{S}et$.

- To define α_Σ: $\mathsf{Pr}^{\mathcal{L}_1}(\Sigma) \to \mathsf{Pr}^{\mathcal{L}_2}(\Phi(\Sigma))$, first note that $\Phi(\Sigma) = \Sigma$, and remember that the objects and morphisms of $\mathsf{Pr}^{\mathcal{L}_i}$ are equivalence classes $[t]$ of terms t over $\mathcal{L}_i$. But since the $\mathsf{Sen}^{\mathcal{DFOL}}(\sigma)$-images of $\mathcal{L}_1$-axioms are consequences of the $\mathcal{L}_2$-axioms, all elements of such an equivalence class of $\mathcal{L}_1$ are mapped to the same equivalence class in $\mathcal{L}_2$. Therefore, for an object or morphism $[t]$ of $\mathsf{Pr}^{\mathcal{L}_1}(\Sigma)$, α_Σ is well-defined by putting $\alpha_\Sigma([t]) = [\mathsf{Sen}^{\mathcal{DFOL}}(\sigma)(t)]$. α_Σ preserves the preordering on $\mathsf{Pr}^{\mathcal{L}_1}(\Sigma)$-morphisms because the preordering is defined by rewrite axioms of $\mathcal{L}_1$, which $\mathsf{Sen}^{\mathcal{DFOL}}(\sigma)$ maps to consequences of $\mathcal{L}_2$-axioms.

- β_Σ: $\mathsf{Mod}^{\mathcal{L}_2}(\Phi(\Sigma)) \to \mathsf{Mod}^{\mathcal{L}_1}(\Sigma)$ is obtained as follows. Define $\mathcal{L}_i^*$ by adding the elements of Σ as constants of sort Ob to $\mathcal{L}_i$; then define σ^* by extending σ such that the new constants are mapped to themselves. Clearly, σ^* is a theory-morphism from $\mathcal{L}_1^*$ to $\mathcal{L}_2^*$. Then a model $(A',m') \in |\mathsf{Mod}^{\mathcal{L}_2}(\Sigma)|$ induces a model $M' \in |\mathsf{Mod}^{\mathcal{DFOL}}(\mathcal{L}_2^*)(\Phi(\Sigma))|$; M' is obtained by taking A' and interpreting each of the new constants $c \in \Sigma$ as $m'(c)$. Then the model reduction of $\mathcal{DFOL}$ yields a model $M \in |\mathsf{Mod}^{\mathcal{DFOL}}(\mathcal{L}_1^*)|$, i.e., $M = \mathsf{Mod}^{\mathcal{DFOL}}(\sigma^*)(M')$. Finally, M induces a model $(A,m) \in |\mathsf{Mod}^{\mathcal{L}_1}(\Sigma)|$; A arises from M by forgetting the interpretations of the new constants, and m is the interpretation of the new constants in M. Thus we put $\beta_\Sigma(A',m') = (A,m)$.
  The action of β_Σ on morphisms is defined similarly.

- The proof of the satisfaction condition is straightforward. □

Proposition 4.16. *Let $\mathcal{L}$ be a categorical logic. There are preorder-enriched institution comorphisms from $\mathcal{I}(\mathcal{L})$ to $\overline{\mathcal{I}(\mathcal{L})}$ and from $\overline{\mathcal{I}(\mathcal{L})}$ to $\mathcal{I}_{thin}(\mathcal{L})$. In particular, semantic consequence is the same in the three institutions.*

Proof : The comorphisms $\mathcal{I}(\mathcal{L}) \to \overline{\mathcal{I}(\mathcal{L})}$ and $\overline{\mathcal{I}(\mathcal{L})} \to \mathcal{I}_{thin}(\mathcal{L})$ are obtained if Prop. 4.15 is applied to the obvious theory morphisms $\mathcal{L} \to \overline{\mathcal{L}}$ and $\overline{\mathcal{L}} \to \mathcal{I}_{thin}(\mathcal{L})$, respectively.

Because the translation of signatures and sentences of the comorphisms is the identity, the claim that semantic consequence coincides is well-defined. And it is easy to prove: Satisfaction is defined via the existence of certain morphisms and the model translations of the comorphisms only identify existing morphisms. □

5 Example

Our framework permits the use of $\mathcal{DFOL}$-theory morphisms to obtain a precise semantics for parametric and modular specifications. Fig. 4 gives a hierarchy of several examples of modular categorical logic specifications. Nodes are categorical logics, the solid arrows are $\mathcal{DFOL}$-theory inclusions, and all rectangles are pushouts.

The pushout of two $\mathcal{DFOL}$-theory inclusions is easy to construct. A theory inclusion $\sigma_i : T \to T_i$ arises by adding declarations D_i and axioms A_i to T for $i = 1, 2$. Then a pushout of σ_1 and σ_1 arises by adding declarations D_1 and D_2 and axioms A_1 and A_2 to T. Since $\mathcal{DFOL}$ is realized as a signature in the logical framework LF[23], this is a special case of the construction given in [24] for LF.

By Prop. 4.2 and Prop. 4.15, all nodes and edges of Fig. 4 induce preorder-enriched institutions and comorphisms between them.

For example, we obtain classical $S4$ as a pushout of classical logic *Prop* and intuitionistic modal logic $IS4$ over intuitionistic logic *IProp*.

All specifications are available in the input syntax of Twelf, an implementation of LF[31], such that they can be type-checked automatically. They can be obtained at [33].

Furthermore, we could reuse a specification for monoidal comonads in the specification of both modal and linear logic: The dotted arrows are theory morphisms that are not inclusions but instantiations. However, a module system for Twelf that could handle these instantiations is not implemented yet.

Example 5.1 (Cartesian closed logic). *CartClosed* specifies implication $\to$ as an exponential object by adding to $Cartesian_{top}$ the declarations
$\to : Ob \times Ob \to Ob.$
$\mathsf{eval} : \Pi B, C : Ob.\ Mor((B{\to}C){*}B, C).$
$\mathrm{curry} : \Pi A, B, C : Ob.\ Mor(A{*}B, C) \to Mor(A, B{\to}C).$
and the axioms
$\forall f : Mor(A{*}B, C).\ \langle \pi_1; \mathsf{curry}(f), \pi_2; id(B)\rangle; \mathsf{eval}(B, C) \leadsto f.$
$\forall f : Mor(A, B \longrightarrow C).\ \mathsf{curry}(\langle \pi_1; f, \pi_2; id(B)\rangle; \mathsf{eval}(B, C)) \leadsto f.$

Example 5.2 (Intuitionistic logic). The theory *Cocartesian* specifies disjunction $+$ as a coproduct and falsity $\bot$ as an initial object by adding to *CatLog* the declarations
$\bot : Ob.$
$!! : Mor(\bot, A).$
$+ : Ob \times Ob \to Ob.$
$\mathsf{inl} : \Pi A, B : Ob.\ Mor(A, A{+}B).$

$\mathsf{inr} : \Pi A, B : Ob.\, Mor(B, A{+}B)$.
$[\_, \_] : Mor(A, C) \times Mor(B, C) \to Mor(A{+}B, C)$.
and the axioms
$\forall f : Mor(\bot, A).\, f \rightsquigarrow !!$.
$\forall f : Mor(A{+}B, C).\, [\mathsf{inl}(A, B); f), \mathsf{inr}(A, B); f] \rightsquigarrow f$.
$\forall f : Mor(A, C)\ \forall g : Mor(B, C).\, \mathsf{inl}(A, B); [f, g] \rightsquigarrow f$.
$\forall f : Mor(A, C)\ \forall g : Mor(B, C).\, \mathsf{inr}(A, B); [f, g] \rightsquigarrow g$.

Combining *Cartesian* and *Cocartesian* by a pushout yields *Bicartesian*, the theory of bicartesian closed categories[17]. To obtain intuitionistic logic *IProp*, we take the pushout of *Bicartesian* and *CartClosed* (as morphisms out of *Cartesian*). Then we only need to add the abbreviation $-A := A \to \bot$ to define negation. In *IProp*, there are already terms of the sorts $Mor(A* - A, \bot)$ and $Mor(A, - - A)$.

Example 5.3 (Classical logic). Several possibilities exist to extend *IProp* to classical logic (see e.g., [17]). We define *Prop* by adding an operation $\mathsf{tnd} : \Pi A : Ob.\, Mor(\top, A + - A)$. We also add rewrite axioms that make proofs from A to A via $- - A$ reducible to $id(A)$.

Example 5.4 (Intuitionistic S4). Following [6], we extend *IProp* to $IS4$ by adding operations $\Box, \Diamond : Ob \to Ob$ and other operations on morphisms modeling the necessity and possibility operators. The necessity modality $\Box$ is interpreted as a monoidal comonad, while the possibility modality $\Diamond$ is interpreted as a monad, which is strong relative to the necessity comonad.

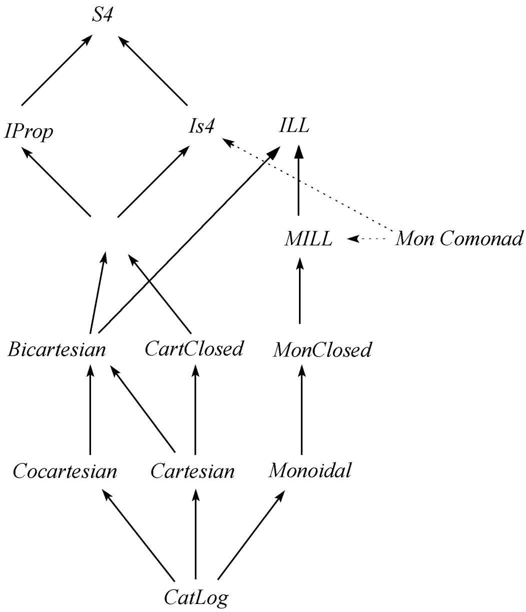

Figure 4. Graph of Logics

In all the above examples, we have the axiom $Des == \top$, which expresses that $\top$ is the minimal designated truth value. Since $\top$ is also a terminal element, the models have (up to isomorphism) the greatest truth value as the only designated one.

Example 5.5 (Linear logic). For a different kind of categorical logic, which does not build up from *CartClosed*, but simply from *CatLog*, consider multiplicative intuitionistic linear logic *MILL* as in [4]. Multiplicative conjunction and linear implication are interpreted as a symmetric monoidal closed category. The of-course operator ! is interpreted as a comonad, and each object $!A$ is equipped with a comonoid structure such that the comonoid maps are also coalgebra maps. We add the axiom $Des == I$, i.e., all truth values greater than the multiplicative truth I are designated. If we take the pushout of *MILL* and *Bicartesian* over *CatLog*, we obtain intuitionistic linear logic *ILL* where the bicartesian structure corresponds to the additive connectives. Note that in *ILL*, we have a morphism from I to $\top$, i.e., $\top$ is a greater designated truth value than I, but not (except for trivial models) the other way round.

All preorder-enriched institutions induced in the above examples are strongly sound by Prop. 4.4. Prop. 4.7 permits the introduction of falsity and disjunction, which yields the deduction theorem for *IProp* and *Prop*. Thus *IProp* and *Prop* are strongly complete. The deduction theorem does not hold for modal logic: For every proof term $x{:}Mor(\top, A)$ there is a proof term $\Box x{:}Mor(\Box\top, \Box A)$, which, together with the canonical morphism $m\_\top{:}Mor(\top, \Box\top)$, yields a morphism $m\_\top;\Box x{:}Mor(\top, \Box A)$, but there is in general no closed proof term of the sort $Mor(A, \Box A)$. A similar argument disproves the deduction theorem for linear logic. Indeed, these preorder-enriched institutions are only weakly complete.

For modal and linear logic, it is not trivial to define the rewriting structure in a way that permits normalization. Therefore, we give most properties for these as equalities instead of as rewrites.

6 Equivalence between λ and *Cat*

$\mathbf{CPL}^{ND\lambda}$ models are valuations into the boolean algebra $\{0, 1\}$ whereas $\mathcal{I}(Prop)$-models are valuations into arbitrary boolean algebras. This difference is not trivial. In a boolean algebra, regarded as a *Prop*-category, an object has a global element iff it is terminal (i.e., equal to the 1 of the boolean algebra). There is no mapping from valuations into arbitrary boolean algebras to valuations into *Bool* that preserves and reflects truth (i.e. terminalhood) of formulas: Let $\nu{:}\{p, q\} \to Bool^2$ map p to $(\top, \bot)$ and q to $(\bot, \top)$. Assume that $\nu'{:}\{p, q\} \to Bool$ makes true the same formulas as ν. Then $\nu'(p)$ and $\nu'(q)$ must both be $\bot$, because neither $\nu(p)$ nor $\nu(q)$ is terminal. However, $\nu(p \vee q)$ *is* terminal, while $\nu'(p \vee q)$ is not, a contradiction. Hence, the model translation of the comorphism from $\mathcal{I}(\mathcal{L})$ to $\overline{\mathcal{I}(\mathcal{L})}$ introduced in Prop. 4.16 cannot be reversed.

A related observation is that $\mathbf{CPL}^{ND\lambda}$ and $\mathcal{I}(Prop)$ differ in their behavior of disjunction. While in $\mathbf{CPL}^{ND\lambda}$, a model M satisfies a disjunction iff it satisfies either of the disjuncts (which is called "model-theoretic disjunction" in [30], this is not the case for $\mathcal{I}(Prop)$: Consider the weakly free model over the theory $(\{A, B\}, \{A{+}B\})$, existing by Prop. 4.14.

However, from the point of view of semantic consequence, we can restrict ourselves to valuations into *Bool*: Semantic consequence in $\mathbf{CPL}^{ND\lambda}$ and $\mathcal{I}(Prop)$ coincide. We turn this observation into a general notion:

Definetion 6.1. Let $\mathcal{L}$ be a categorical logic, and let $A \in |\overline{\mathbb{C}(\mathcal{L})}|$. Let $\mathcal{I}^A(\mathcal{L})$ denote the institution obtained from $\mathcal{I}(\mathcal{L})$ by allowing only those Σ-models (M, m) for which $M = A$. We denote satisfaction and semantic consequence in this institution

with the superscript A. The category A is called a *weak truth value object* for $\mathcal{L}$ if $\varphi \models^A \psi$ implies $\varphi \models \psi$ for all sentences φ, ψ, and a *strong truth value object* if $\Phi \models^A \psi$ implies $\Phi \models \psi$ for all sets Φ of sentences and sentences ψ.
(N.B.: Even in the presence of conjunction, a weak truth value object need not be a strong truth value object, as the set Φ in question may be infinite.)

Defination 6.2. Let $\mathcal{L}$ *be a categorical logic extending the theory of cartesian closed categories (i.e., minimal intuitionistic logic with* $\top$, $*$, $\rightarrow$*), and let* $A \in |\overline{\mathbb{C}(\mathcal{L})}|$ *be thin and skeletal, hence essentially being a partial ordering* $\leqslant$*. Then* A *is a strong truth value object for* $\mathcal{L}$ *iff the following condition holds.*

($*$) *Let* $B \in |\overline{\mathbb{C}(\mathcal{L})}|$*, and let* $a, b \in |B|$*. If* $f(a) \leqslant f(b)$ *for each* $\mathcal{L}$*-morphism* $f : B \rightarrow A$*, then* $\hom_B(a, b) \neq \emptyset$.

(If B is a thin category, i.e. a preorder, then condition ($*$) states that the source of all morphisms from B into A is jointly order-reflecting.)

Proof: Assume that ($*$) holds, let $\Phi \models^A \psi$, and let $(B, m) \models \Phi$. Then for every $f : B \rightarrow A$, $(A, f \circ m) \models \Phi$ and hence $(A, f \circ m) \models \psi$, i.e., $\top \leqslant f(m(\psi))$. By ($*$), it follows that $B(\top, m(\psi)) \neq \emptyset$, i.e., $(B, m) \models \psi$.

Conversely, let A be a strong truth value object, let $B \in |\overline{\mathbb{C}(\mathcal{L})}|$, and let $a, b \in |B|$ such that $A(f(a), f(b)) \neq \emptyset$ for all $f : B \rightarrow A$. Let $\Sigma = |B|$, and let Φ be the theory (i.e., the set of valid formulas) of the Σ-model (B, η) where η: $\Sigma \rightarrow |B|$ is the inclusion. Since morphisms $f : B \rightarrow A$ are then just the Σ-valuations in A that validate Φ (because by the assumption on $\mathcal{L}$, formulas χ_1 and χ_2 denote isomorphic objects of B iff $(B, \eta) \models \chi_1 \longleftrightarrow \chi_2$), the premise says that $\Phi \models^A a \rightarrow b$ (note that the assumption on $\mathcal{L}$ implies that an implication holds iff there exists a morphism between the corresponding objects). Thus, $\Phi \models a \rightarrow b$, so that $a \rightarrow b$ holds in (B, η); i.e., $B(a, b) \neq \emptyset$ as claimed. □

Example 6.3. *Bool*, seen as a *Prop*-category, is a strong truth value object for *Prop*. For any dense-in-itself metric space X, the Heyting algebra $\mathfrak{O}(X)$ of open sets in X (qua bicartesian closed category) is a weak truth value object for *IProp*[34].

One can use the above result to show that no strong truth value object for *IProp* exists. To see this, assume that A is a Heyting algebra satisfying condition ($*$) of Proposition 6.2. Let $\alpha > |A|$ be a cardinal, and let B be the ordinal $\alpha + 2$, considered as a Heyting algebra. The two largest elements of $\alpha + 2$ are α and $\alpha + 1$. We show that for every morphism $f : B \rightarrow A$, $f(\alpha) = f(\alpha + 1)$; then by ($*$), $\alpha + 1 \leqslant \alpha$, contradiction. For cardinality reasons, we have $a < b$ in B such that $f(a) = f(b)$. Then

$$f(a) = f(b \rightarrow a) = (f(b) \rightarrow f(a)) = \top = f(\alpha + 1),$$

and since $a \leqslant \alpha$, we obtain $f(\alpha) = f(\alpha + 1)$ as claimed.

A consequence of this observation is that, for every dense-in-itself metric space X, the consequence relation on $\mathfrak{O}(X)$ is non-compact, since otherwise, the weak truth value object $\mathfrak{O}(X)$ would also be a strong truth value object.

We are now ready to formalize the Curry-Howard-Tait correspondence in terms of preorder-enriched institutions. We cannot expect this to be constructed in an institution-independent manner. Rather, for some given categorical propositional logic $\mathcal{L}$, we can relate $\mathcal{I}(\mathcal{L})$ to a well-known institution. We follow on the five stages in Fig. 2 and map the natural deduction logics to categorical logics as depicted in Fig. 5. [4]

void logic	*CatLog*
conjunctive logic	*Cartesian*
conjunctive-implicational logic	*CartClosed*
intuitionistic logic	*IProp*
classical logic	*Prop*

Figure 5. Five stages of the correspondence

Figure 6 describes the mapping from categorical logic to natural deduction. A morphism $M : Mor(A, B)$ in categorical logic is mapped to a natural deduction proof term $\alpha_x(M)$ of form $x : A \rhd \alpha_x(M) : B$. Note that $\alpha_x(M)$ may contain the free variable $x : A$. For the top-level translation, we may chose x in an arbitrary way.

Figure 7 describes the converse mapping from natural deduction to categorical logic. Since the rule system in Fig. 2 deals with multi-variable contexts, we translate a proof $x_1 : A_1, \ldots x_n : A_n \rhd M : B$ to a morphism $\alpha'(M) : Mor(A_1 \times \cdots \times A_n B)$.

M	$x \rhd \alpha_x(M)$
$id : Mor(A, A)$	$x : A \rhd x : A$
$(M : Mor(A, B)); (N : Mor(B, C))$	$x : A \rhd \alpha_y(N)[y := \alpha_x(M)]$
$! : Mor(A, \top)$	$x : A \rhd \Delta : \top$
$\langle M, N\rangle : Mor(A, B \times C)$	$x : A \rhd \langle \alpha_x(M), \alpha_x(N)\rangle : B \wedge C$
$\pi_1 : Mor(A \times B, A)$	$x : A \wedge B \rhd \mathsf{fst}(x) : A$
$\pi_2 : Mor(A \times B, A)$	$x : A \wedge B \rhd \mathsf{snd}(x) : B$
$\mathsf{eval} : Mor(B^A \times A, B)$	$x : (A \rightarrow B) \wedge A \rhd \mathsf{fst}(x)\,\mathsf{snd}(x) \;:\; B$
$\mathsf{curry}(M) : Mor(A, C^B)$	$x : A \rhd \lambda y{:}B \,.\, \alpha_z(M)[z := \langle x, y\rangle]$
$!! : Mor(\bot, A)$	$x : \bot \rhd \nabla_A(x) : A$
$\mathsf{inl} : Mor(A, A + B)$	$x : A \rhd \mathsf{inl}(x) : A \vee B$
$\mathsf{inr} : Mor(B, A + B)$	$x : B \rhd \mathsf{inr}(x) : A \vee B$
$[M, N] : Mor(A + B, C)$	$x : A \vee B \rhd \mathsf{case}\ x\ \mathsf{of}\ \mathsf{inl}(y) \rightarrow \alpha_y(M) \mid \mathsf{inr}(z) \rightarrow \alpha_z(N)$
$\mathsf{tnd} : Mor(\top, A + (A \rightarrow \bot))$	$x : \top \rhd \mathsf{tnd}_A : A \vee \neg A$

Figure 6. Mapping categorical logic to natural deduction, following the five stages in Fig.5

M	$\alpha'(M)$
$\Gamma, x : A \rhd x : A$	$\pi_i : Mor(\prod \Gamma \times A, A)$ resp. $id : Mor(A, A)$
$\Gamma \rhd \Delta : \top$	$! : Mor(\prod \Gamma, \top)$
$\Gamma \rhd \langle M, N\rangle : A \wedge B$	$\langle \alpha'(M), \alpha'(N)\rangle : Mor(\prod \Gamma, A \times B)$
$\Gamma \rhd \mathsf{fst}(M) : A$	$\alpha'(M); \pi_1 : Mor(\prod \Gamma, A)$
$\Gamma \rhd \mathsf{snd}(M) : B$	$\alpha'(M); \pi_2 : Mor(\prod \Gamma, B)$
$\Gamma \rhd \lambda x{:}A.M : A \rightarrow B$	$\mathsf{curry}(\alpha'(M)) : Mor(\prod \Gamma \times A, B)$
$\Gamma \rhd MN : B$	$\langle \alpha'(M), \alpha'(N)\rangle; \mathsf{eval} : Mor(\prod \Gamma, B)$
$\Gamma \rhd \nabla_A(M) : A$	$\alpha'(M); !! : Mor(\prod \Gamma, A)$
$\Gamma \rhd \mathsf{inl}(M) : A \vee B$	$\alpha'(M); \mathsf{inl} : Mor(\prod \Gamma, A + B)$
$\Gamma \rhd \mathsf{inr}(M) : A \vee B$	$\alpha'(M); \mathsf{inr} : Mor(\prod \Gamma, A + B)$
$\Gamma \rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \rightarrow N \mid \mathsf{inr}(y) \rightarrow P \;:\; C$	$\langle \alpha'(M), id\rangle; [\mathsf{curry}(\alpha'(N)), \mathsf{curry}(\alpha'(P))]; \mathsf{eval}$ $: Mor(\prod \Gamma, C)$
$\Gamma \rhd \mathsf{tnd}_A : A \vee \neg A$	$!; \mathsf{tnd} : Mor(\prod \Gamma, A + (A \rightarrow \bot))$

Figure 7. Mapping natural deduction to categorical logic, following the five stages in Fig.5. π_i is the i-th projection, obtained as a composite of π_1 and π_2

[4] See [33] for the full specifications of the categorical logics.

This corresponds to a switch from multi-variable to one-variable contexts using conjunction. Ultimately, we need this translation only for one-variable contexts on both sides; however, the recursive definition at some places involves enlargement of contexts.

$$
\begin{array}{lll}
\langle fst(M), snd(M)\rangle & \rightsquigarrow_\eta & M \\
\lambda x.Mx & \rightsquigarrow_\eta & M \\
x:\bot \rhd M : A & \rightsquigarrow_\eta & \nabla_A(x) \\
case x of \mathsf{inl}(y) \longrightarrow N[x := \mathsf{inl}(y)] y N[x := \mathsf{inr}(y)] & \rightsquigarrow_\eta & N
\end{array}
$$

Figure 8. η-reduction rules for proof terms

Example 6.4. Assume that the η-rules of Fig. 8 are added to the reduction of $\mathbf{CPL}^{ND\lambda}$. Moreover, let $\overline{\mathbf{CPL}}^{ND\lambda}$ be $\mathbf{CPL}^{ND\lambda}$ with proof reductions quotiented out. Then, for any signature $\Sigma \in \mathbb{S}et$,

1. the mapping given in Fig. 6 is a preorder-preserving functor α_Σ: $\mathsf{Sen}^{Prop}(\Sigma) \longrightarrow \mathsf{Sen}^{\mathbf{CPL}^{ND\lambda}}(\Sigma)$, and similarly for the other stages in Fig. 5.
2. the mapping given in Fig. 7 is a functor α'_Σ:$\mathsf{Sen}^{\overline{\mathbf{CPL}}^{\mathsf{ND}\lambda}}(\Sigma) Sen^{\overline{Prop}}(\Sigma)$, and similarly for the other stages in Fig. 5.
3. α_Σ naturally extends to quotienting of proofs with respect to the reduction relation, and then becomes the inverse of α'_Σ (hence, both are isomorphisms).

Proof: 1. Preservation of identity and composition follows from the first two lines in the table of Fig. 6. Preservation of proof reduction is proved by considering all the reduction axioms. For example, consider the reduction axiom $\langle f;\ \pi_1, f;\ \pi_2\rangle \rightsquigarrow f$. We have $\alpha(\langle f;\pi_1, f;\pi_2\rangle) = \langle \mathsf{fst}(\alpha(f)), \mathsf{snd}(\alpha(f))\rangle \rightsquigarrow_\eta \alpha(f)$.

2. Preservation of identities is clear. Preservation of composition follows from the fact that

$$(\alpha'(x : A \rhd M : B) \times id); \alpha'(y : B, \Gamma \rhd N : C) \quad \sim \quad \alpha'(x : A, \Gamma \rhd N[y := M]),$$

where $\sim$ is the equivalence relation generated by reducibility. This fact is proved by induction over N. For example, assume that by induction hypothesis $\alpha'(y : B, z : D, \Gamma \rhd N[y := M]) \sim (\alpha'(M) \times id); \alpha'(N)$. Assuming suitable α-renaming, then also

$$
\begin{array}{l}
\alpha'(y : B, \Gamma \rhd (\lambda z : D.N)[y := M]) \ = \\
\alpha'(y : B, \Gamma \rhd \lambda z : D.(N[y := M])) \ = \\
\mathsf{curry}(\alpha'(N[y := M])) \ \sim \\
\mathsf{curry}((\alpha'(M) \times id); \alpha'(N)) \ \rightsquigarrow \\
\alpha'(M); \mathsf{curry}(\alpha'(N))
\end{array}
$$

We should remark that most of the cases even go through with $\rightsquigarrow$ instead of $\sim$, which would show α'_Σ to be a lax functor. The only problem is the treatment of variables.

Here we have

$$\begin{aligned}&(\alpha'(x : A \rhd M : B) \times id); \pi_1 \ = \\ &\langle \pi_1; \alpha'(x : A \rhd M : B), \pi_2\rangle; \pi_1 \ \rightsquigarrow \\ &\pi_1; \alpha'(x : A \rhd M : B) \ = \\ &\alpha'(x : A, \Gamma \rhd M : B)\end{aligned}$$

but for obtaining a lax functor, the opposite rewrite would be needed. This point needs further investigation.

3. Again by induction, one can show that $\alpha_\Sigma(\alpha'_\Sigma(M))$ is equivalent to M, and $\alpha'_\Sigma(\alpha_\Sigma(N))$ is equivalent to N. □

Theorem 6.5. There is an isomorphism $\mathcal{I}^{Bool}(\overline{Prop})$ to $\overline{\mathbf{CPL}}^{ND\lambda}$.

Proof: The signature translations are the identity functors; the model translations are the obvious bijections; and the sentence translation is given by Prop. 6.4. □

Similarly, there is an isomorphism between $\overline{\mathcal{I}(IProp)}$ and intuitionistic logic with Heyting algebra semantics. For modal and to a lesser extent for linear logic, the Curry-Howard-Tait correspondence is not so much a proved result, but rather a design paradigm that is satisfied a priori. In these cases, it makes more sense to see the institution constructed within our framework as the incarnation of the correspondence.

7 Conclusion and Future Work

We have presented a canonical way of obtaining proof-theoretic institutions for categorical propositional logics, following the spirit of the Curry-Howard-Tait isomorphism. We have proved generic deduction, soundness and completeness theorems, and given examples of categorical logics, for which categorical treatment had already been established in a non-institutional framework. Our definitions have the crucial advantage that they offer the possibility of parametric and modular specifications, and that they have been formalized (and machine-checked) as Twelf specifications. We avoid the use of more complicated frameworks like polycategories by relying on conjunction for the treatment of multi-assumption contexts.

For classical logic, the institutional structure sheds light on the usual collapsing of proofs problem in classical logic (classical bicartesian closed categories are boolean algebras), which we avoided by using preorder-enriched categories, as in [17]. For Linear Logic, the specification of the modality ! is considerably involved; to simplify this specification, we might use a different categorical model (see [29]). But this would be too much of a departure from the point of view we have taken in this paper of using established notions of categorical models, not involving fibrations or indexed categories, for the time being. Another interesting direction left to future work is to consider linear logic with both a classical and a linear function space as in [28].

The Curry-Howard-Tait isomorphism can be recovered as an explicit isomorphism between institutions, one institution using proof trees, the other one λ-terms. Concerning the relation between various λ-calculi and categorical logic, only a weaker correspondence could be set up. One obstacle is the difference between, e.g., boolean algebra-valued and *Bool*-valued models. We have provided some general results about the relations between such models. Another obstacle are the different notions of proofs and proof reductions that make it hard to obtain isomorphic reduction-preserving translations. In order to obtain even the weaker correspondences, the usual β-rules for proof terms have to be extended by η-rules. But even then, functoriality of proof translation only holds for one direction; for the other direction, proof reductions have to be quotiented out. Without the quotienting, we were almost able to obtain a lax functor, but there are problems caused by the necessary switching between variable contexts. This point needs further investigation.

The study of further properties, such as Craig interpolation and Beth definability, is the subject of future work, as is the extension of our framework to first-order logic. The latter will require a more powerful meta-language, namely one that permits to declare function symbols that take functions as arguments. Another interesting question is whether the institutions $\mathcal{I}(\mathcal{L})$ have elementary diagrams in the sense of [10]. Our current notion of model is obviously too weak to ensure this; for ensuring elementary diagrams one would need "intensional models" over signatures containing proof variables, to be valuated with proofs — such models would not only determine which propositions are true, but also why. With such models, it should also be easy to show liberality of $\mathcal{I}(\mathcal{L})$.

Acknowledgement

We want to thank Răzvan Diaconescu for useful comments.

References

[1] Adámek J, Herrlich H, Strecker GE. Abstract and Concrete Categories, Wiley Interscience, 1990.

[2] Avron A. The Semantics and Proof Theory of Linear Logic. Technical Report, ECS-LFCS-87-27, University Edinburgh, 1987.

[3] Bellin G, Hyland M, Robinson E, Urban C. Categorical proof theory of classical propositional calculus. Theor. Comput. Sci, 2006, 364(2): 146–165.

[4] Benton N, Bierman G, de Paiva V, Hyland M. Term Assignment for Intuitionistic Linear Logic. Technical Report, 262, University of Cambridge Computer Laboratory, 1992.

[5] Bidoit M, Hennicker R. On the integration of observability and reachability concepts. In Fossacs 2002, LNCS 2303, pp. 21–36. Springer-Verlag, 2002.

[6] Bierman G, de Paiva V. On an intuitionistic modal logic. Studia Logica, 2000, 65: 383–416.

[7] Borzyszkowski T. Moving specification structures between logical systems. In WADT 98, LNCS 1589, pp. 16–30, Springer-Verlag, 1999.

[8] Cîrstea C. Institutionalising many-sorted coalgebraic modal logic. In Coalgebraic Methods in Computer Science, ENTCS65, Elsevier, 2002.

[9] Diaconescu R. Institution-independent ultraproducts. Fundamenta Informaticae, 2003, 55: 321–348.

[10] Diaconescu R. Elementary diagrams in institutions. J. Logic Computation, 2004, 14(5): 651–674.

[11] Diaconescu R. Herbrand theorems in arbitrary institutions. Information Processing Letters, 2004, 90: 29–37.

[12] Diaconescu R. An institution-independent proof of Craig Interpolation Theorem. Studia Logica, 2004, 77(1): 59–79.

[13] Diaconescu R. Institution-independent model theory. Birkhäuser Basel, 2008. To appear.

[14] Diaconescu R, Goguen J, Stefaneas P. Logical support for modularisation. Logical Environments, pp. 83–130. Cambridge, 1993.

[15] Došen K. Cut Elimination in Categories. Kluwer Academic Publishers, Dordrecht, 1999.

[16] Fiadeiro JL, Costa JF. Mirror, mirror in my hand: A duality between specifications and models of process behaviour. Math. Struct. Comput. Sci., 1996, 6(4): 353–373.

[17] Führmann CF, Pym D. On the geometry of interaction for classical logic. LICS 2004, pp. 211–220. IEEE Computer Society Press, 2004. Extended version at http://www.cs.bath.ac.uk/~cf/dj.pdf

[18] Gallier JH. Constructive logics part I: A tutorial on proof systems and typed gamma-calculi. Theor. Comput. Sci., 1993, 110(2): 249–339.

[19] Goguen JA, Burstall RM. A study in the foundations of programming methodology: Specifications, institutions, charters and parchments. In: D. P. et al., eds, Category Theory and Computer Programming, LNCS 240, pp.313–333. Springer-Verlag, 1985.

[20] Goguen JA, Burstall RM. Institutions: Abstract model theory for specification and programming. J. ACM, 1992, 39: 95–146.

[21] Goguen JA, Diaconescu R. Towards an algebraic semantics for the object paradigm. In WADT 94, LNCS785, Springer-Verlag, 1994.

[22] Goguen JA, Roşu G. Institution morphisms. Formal Aspects of Computing, 2002, 13: 274–307.

[23] Harper R, Honsell F, Plotkin G. A framework for defining logics. Journal of the Association for Computing Machinery, 1993, 40(1): 143–184.

[24] Harper R, Sannella D, Tarlecki A. Structured presentations and logic representations. Annals of Pure and Applied Logic, 1994, 67: 113–160.

[25] Howard WA. The formulas-as-types notion of construction. In: Seldin JP, Hindley JR, eds, To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism, pp. 479–490. Academic Press, 1980.

[26] Lambek J, Scott PJ. Introduction to Higher Order Categorical Logic. Cambridge, 1986.

[27] Mac Lane S. Categories for the Working Mathematician. Springer-Verlag, 1997.

[28] Maietti M, de Paiva V, Ritter E. Categorical Models for Intuitionistic and Linear Type Theory. In: Tiuryn J, ed. Foundations of Software Science and Computation Structures, Third International Conference, pp. 223–237. Springer-Verlag, 2000.

[29] Maietti ME, Maneggia P, de Paiva V, Ritter E. Relating categorical semantics for intuitionistic linear logic. Appl. Categ. Structures, 2005, 13(1): 1–36.

[30] Mossakowski T, Goguen J, Diaconescu R, Tarlecki A. What is a logic? In: Beziau JY, ed. Logica Universalis, pp. 113–133. Birkhäuser, 2005.

[31] Pfenning F, Schürmann C. System description: Twelf - a meta-logical framework for deductive systems. LNCS 1632, pp.202–206, 1999.

[32] Rabe F. First-Order Logic with Dependent Types. In: Shankar N, Furbach U, eds. Proceedings of the 3rd International Joint Conference on Automated Reasoning, LNCS 4130, pp. 377–391, Springer-Verlag, 2006.

[33] Rabe F, de Paiva V, Mossakowski T. Categorical Logic Specifications, 2007. http://kwarc.eecs.iu-bremen.de/frabe/Research/cht/index.html.

[34] Rasiowa H, Sikorski R. The Mathematics of Metamathematics. PWN, Warsaw, 1963.

[35] Sannella D, Tarlecki A. Specifications in an arbitrary institution. Inform. and Comput., 1988, 76: 165–210.

[36] Schröder L, Mossakowski T, Lüth C. Type class polymorphism in an institutional framework. In: Fiadeiro J, ed. WADT 04, LNCS 3423, pp.234–248, Springer-Verlag, 2004.

[37] Seely RAG. Hyperdoctrines, natural deduction and the Beck condition. Zeitschrift für mathematische Logik und Grundlagen der Mathematik, 1983, 29(6): 505–542.

[38] Seely RAG. Locally cartesian closed categories and type theory. Mathematical Proceedings of the Cambridge Philosophical Society, 1984, 95(1): 33–48.

[39] Seely RAG. Linear logic, *-autonomous categories, and cofree coalgebras. Categories in Computer Science and Logic, Contemp. Math., pp.371–382. AMS, 1989.

[40] Sernadas A, Sernadas C, Caleiro C, Mossakowski T. Categorical fibring of logics with terms and binding operators. In: Gabbay D, Rijke MD, eds. Frontiers of Combining Systems 2, Studies in Logic and Computation, pp.295–316. Research Studies Press, 2000.

[41] Szabo ME. Polycategories. Communications in Algebra, 1975, 3(8): 663–689.

[42] Tarlecki A. On the existence of free models in abstract algebraic institutions. Theoret. Comput. Sci., 1985, 37(3): 269–304.